Derivation of the 4 point frequency formula

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If we assume uniform sampling and we have 4 consecutive samples, \( y_0, y_1, y_2, \) and \( y_3 \). Then if we assume they are consecutive samples of a sinusoid with a DC offset, then we may write them as:

\[
y_0 = \mu + A \cos(\varphi - 3\theta/2) = \mu + A \cos(\varphi) \cos(3\theta/2) - A \sin(\varphi) \sin(3\theta/2) \tag{1}
\]

\[
y_1 = \mu + A \cos(\varphi - \theta/2) = \mu + A \cos(\varphi) \cos(\theta/2) - A \sin(\varphi) \sin(\theta/2) \tag{2}
\]

\[
y_2 = \mu + A \cos(\varphi + \theta/2) = \mu + A \cos(\varphi) \cos(\theta/2) + A \sin(\varphi) \sin(\theta/2) \tag{3}
\]

\[
y_3 = \mu + A \cos(\varphi + 3\theta/2) = \mu + A \cos(\varphi) \cos(3\theta/2) + A \sin(\varphi) \sin(3\theta/2) \tag{4}
\]

And here \( A \) is the amplitude, \( \varphi \) is an arbitrary phase, \( \theta \) is the angular step per sample and \( \mu \) is the mean offset.

Since our angular step, \( \theta = \frac{2\pi}{f_s} \), then we need to find theta to solve for \( f \).

So let’s find the pair of differences \([4]-[1]\) and \([3]-[2]\), thus

\[
y_3 - y_0 = 2 \cdot A \cdot \sin(\varphi) \sin(3\theta/2) \tag{5}
\]

\[
y_2 - y_1 = 2 \cdot A \cdot \sin(\varphi) \sin(\theta/2) \tag{6}
\]

Now find the ratio of \([5]/[6]\),

\[
\frac{y_3 - y_0}{y_2 - y_1} = \frac{\sin(3\theta/2)}{\sin(\theta/2)} = \frac{4 \cdot \cos^2(\theta/2) - 1 \cdot \sin(\theta/2)}{\sin(\theta/2)} = 4 \cdot \cos^2(\theta/2) - 1 = 2 \cdot \cos(\theta) + 1 \tag{7}
\]

So

\[
\theta = \cos^{-1} \left[ \frac{1}{2} \left( \frac{y_0 - y_3}{y_2 - y_1} \right) \right] \tag{8}
\]

So finally

\[
f = \frac{f_s}{2\pi} \theta = \frac{f_s}{2\pi} \cos^{-1} \left[ \frac{1}{2} \left( \frac{y_3 - y_0}{y_2 - y_1} - 1 \right) \right] \tag{9}
\]