Derivation of the 4 point frequency formula

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If we assume uniform sampling and we have 4 consecutive samples, y_0, y_1, y_2, and y_3. Then if we assume they are consecutive samples of a sinusoid with a DC offset, then we may write them as:

$$y_0 = \mu + A\cos(\varphi - 3\theta/2) = \mu + A\cos(\varphi)\cos(3\theta/2) - A\sin(\varphi)\sin(3\theta/2) \qquad [1]$$

$$y_1 = \mu + A\cos(\varphi - \theta/2) = \mu + A\cos(\varphi)\cos(\theta/2) - A\sin(\varphi)\sin(\theta/2) \qquad [2]$$

$$y_2 = \mu + A\cos(\varphi + \theta/2) = \mu + A\cos(\varphi)\cos(\theta/2) + A\sin(\varphi)\sin(\theta/2) \qquad [3]$$

$$y_3 = \mu + A\cos(\varphi + 3\theta/2) = \mu + A\cos(\varphi)\cos(3\theta/2) + A\sin(\varphi)\sin(3\theta/2) \qquad [4]$$

And here A is the amplitude, φ is an arbitrary phase, θ is the angular step per sample and μ is the mean offset.

Since our angular step, $\theta = 2\pi \frac{f}{f_s}$, then we need to find theta to solve for *f*.

So let's find the pair of differences [4]-[1] and [3]-[2], thus

$$y_3 - y_0 = 2 \cdot A \cdot \sin(\varphi)\sin(3\theta/2) \qquad [5]$$

$$y_2 - y_1 = 2 \cdot A \cdot \sin(\varphi)\sin(\theta/2) \qquad [6]$$

Now find the ratio of [5]/[6],

$$\frac{y_3 - y_0}{y_2 - y_1} = \frac{\sin(3\theta/2)}{\sin(\theta/2)} = \frac{\left(4 \cdot \cos^2(\theta/2) - 1\right) \cdot \sin(\theta/2)}{\sin(\theta/2)} = 4 \cdot \cos^2(\theta/2) - 1 = 2 \cdot \cos(\theta) + 1 \qquad [7]$$

So $\theta = \cos^{-1}\left[\frac{1}{2}\left(\frac{y_3 - y_0}{y_2 - y_1} - 1\right)\right]$ [8]

So finally

$$f = \frac{f_s}{2\pi}\theta = \frac{f_s}{2\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{y_3 - y_0}{y_2 - y_1} - 1\right)\right) \qquad [9]$$