Butterworth Filter Formulae

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Butterworth

For Butterworth filters, normalized to a cutoff of 1 radian per second, the following are the Laplace forms. These are factored into a product of biquads for even order and a product of one linear with biquads for odd orders.

Even Order N:

$$H(s) = \prod_{n=1}^{N/2} \left(\frac{1}{s^2 + \alpha_n s + 1} \right)$$

where

$$\alpha_n = 2\cos\left(\frac{\pi}{2N}(2n-1) \right)$$

Odd Order N (N>1):

$$H(s) = \frac{1}{s+1} \prod_{n=1}^{\frac{N-1}{2}} \frac{1}{s^2 + \alpha_n s + 1}$$

where

$$\alpha_n = 2\cos\left(\frac{n\pi}{N} \right)$$

Odd Order N (N=1)

$$H(s) = \frac{1}{s+1}$$

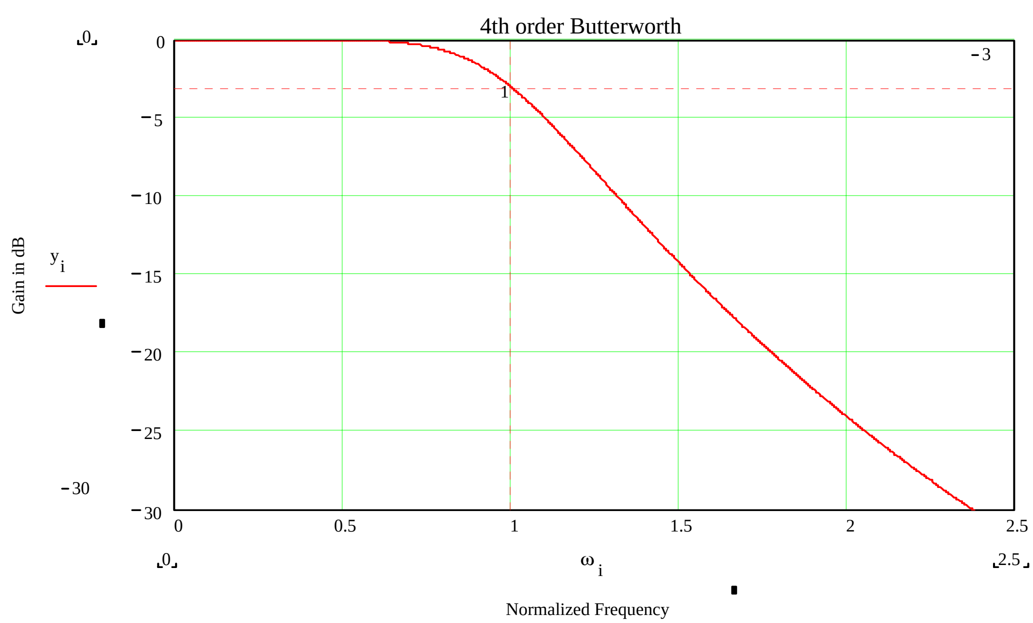
Butterworth Example

For a 4th order Butterworth Filter, we have an even case which factors into 2 biquads. So we find our alphas to be simply $2\cos\left(\pi/8\right)$ and $2\cos\left(3\pi/8\right)$. This yields our transfer equation: $H(s) = \dfrac{1}{s^2+\sqrt{2+\sqrt{2}}s+1} \times \dfrac{1}{s^2+\sqrt{2-\sqrt{2}}s+1}$.

Realizing the numerators are all simply one, we can make a simple table of the denominators for the 1st few orders of Butterworth Filters.

Filter Order	Denominator of Transfer Function
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+\sqrt{2+\sqrt{2}}s+1)(s^2+\sqrt{2-\sqrt{2}}s+1)$
5	$(s+1)\left(s^2+\left(\frac{\sqrt{5}+1}{2}\right)s+1\right)\left(s^2+\left(\frac{\sqrt{5}-1}{2}\right)s+1\right)$

A plot of the 4th Order Butterworth response is shown below:



Bilinear Transform

If one is using s domain filters normalized to 1 radian per second, the standard BLT becomes simply:

$$s = c\frac{1-z^{-1}}{1+z^{-1}}$$

with $c = \cot(\pi f)$ and $f = {}^{f_c}\!/_{f_s}$. Hence f is the normalized digital cutoff frequency for the filter. For example, if a digital filter with a cutoff of 1 kHz is needed and the sampling rate is 8 kHz, then $f = 0.125$ and $c = 1+\sqrt{2}$.

If the above BLT is applied to the standard 1st order low pass filter, we find:

$$\frac{1}{s+1} \Rightarrow \left(\frac{1}{1+c}\right)\left(\frac{1+z^{-1}}{1+\left(\frac{1-c}{1+c}\right)z^{-1}}\right)$$

The above form has a common gain factored out of the numerator.

The standard 2nd order low pass section may be transformed similarly. We find for it:

$$\frac{1}{s^2+\alpha s+1} \Rightarrow \left(\frac{1}{1+c(c+\alpha)}\right)\left(\frac{1+2z^{-1}+z^{-2}}{1+\frac{2(1-c^2)}{1+c(c+\alpha)}z^{-1}+\frac{1+c(c-\alpha)}{1+c(c+\alpha)}z^{-2}}\right)$$