Finding Features of Discretely Sampled Signals

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1 Abstract

This paper presents algorithms for feature extraction. The features in question are maxima, minima and zeroes of discretly sampled signals. The need for this method arises in situations where one does not wish to highly oversample data but still needs the precise locations of the features.

2 Nyquist Theorem

The Nyquist theorem makes an important claim. The claim is that if a continuous bandlimited signal is periodically sampled at a rate of at least two times the bandwidth of the signal, that all of the signal's information is contained within the sampled signal. Thus, if the signal's frequency content is in $[-f_c, f_c]$, then the sampling rate, f_s must obey the following condition $f_s > 2f_c$.

The theorem's correlary states how to reconstruct the unique bandlimited continuous signal from the sampled signal. If the discrete signal is denoted by $f[i]$ and the continuous signal is denoted by $f_c(t)$, and the sampling period is T, then the reconstruction formula is:

$$f_c(t) = \sum_{i=-\infty}^{\infty} f[i] \frac{sin(\pi(t-iT)/T)}{\pi(t-iT)/T)} \tag{1}$$

3 Algorithms

The approach to feature extraction first relies on a simple hunt through the discrete data to locate (within a sample interval) the feature in question. For example, to find the maximum (minimum) just find the biggest (smallest) value of $x[n]$. However, to find a zero, hunt through the data until a consective pair has a negative product, and then use the smaller (in an absolute value way) of the pair's domain value to indicate the zero's location. Now that the feature has been located to within a sampling interval, this location is used as a seed for an iterative process to find the precise location of the feature. To state this in a mathematical way, let the index of the sample nearest the feature be denoted n., Also we will use a normalized sampling period in all of the following formulas,i.e., the sampling period equals one. The refinement in the position of the feature becomes the problem in finding x that satisfies one of the following equations. The first is for maximum or a minimum, and the second is for a zero.

$$f'(n+x) = 0 \tag{2}$$

$$f(n+x) = 0 \tag{3}$$

These are efficiently solved with Newton's iteration. The starting value for x is simply zero. The iteration formulae for solving the two former equations are respectively:

$$x_{j+1} = x_j - \frac{f'(n+x_j)}{f''(n+x_j)} \tag{4}$$

$$x_{j+1} = x_j - \frac{f(n+x_j)}{f'(n+x_j)} \tag{5}$$

4 Interpolation Formulae

If we take (1) and substitute $t = x$ and $T = 1$ and use a trig. identity for the difference of angles, we find:

$$f(x) = \frac{\sin(\pi x)}{\pi} \sum_{i=-\infty}^{\infty} \frac{f[i](-1)^i}{x-i} \tag{6}$$

Now this alternating series may be truncated to make the calculation practical. If the sum is limited to $2\ell + 1$ terms centered around n, the following approximation for $f(n+x)$ is found:

$$f(n+x) \approx \frac{\sin(\pi x)}{\pi} \sum_{i=-\ell}^{\ell} \frac{f[n+i](-1)^i}{x-i} \tag{7}$$

And to make everything complete $f(n+0) = f[n]$. Since x is in the open interval $(-1, 1)$, there is only one integral value for x, namely zero.

For the first and second derivative functions, there are separate formulae for $x = 0$ and $x \neq 0$ cases.

$$f'(n+x) \approx \cos(\pi x) \sum_{i=-\ell}^{\ell} \frac{f[n+i](-1)^i}{x-i} - \frac{\sin(\pi x)}{\pi} \sum_{i=-\ell}^{\ell} \frac{f[n+i](-1)^i}{(x-i)^2} \tag{8}$$

$$f'(n+0) \approx - \sum_{\substack{i=-\ell \\ i \neq 0}}^{\ell} \frac{f[n+i](-1)^i}{i} \tag{9}$$

$$f''(n+x) \approx \frac{\sin(\pi x)}{\pi} \sum_{i=-\ell}^{\ell} f[n+i](-1)^i \left[\frac{2}{(x-i)^3} - \frac{\pi^2}{x-i} \right] - 2\cos(\pi x) \sum_{i=-\ell}^{\ell} \frac{f[n+i](-1)^i}{(x-i)^2} \tag{10}$$

$$f''(n+0) \approx -\frac{\pi^2}{3} f[n] - 2 \sum_{\substack{i=-\ell \\ i \neq 0}}^{\ell} \frac{f[n+i](-1)^i}{i^2} \tag{11}$$

5 References

[1] Oppenheim, Alan V. and Schafer, Ronald W. *Discrete-Time Signal Processing* pp 80-91 Prentice Hall, Englewood Cliffs NJ 1989