Showing Sync() and Rect() form a Fourier Pair

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As is commonly learned in signal processing, the functions Sync() and Rect() form a Fourier pair. And usually the proof for this goes along the lines of taking the Fourier transform of Rect() and getting Sync()[1]. Then for the other way around the properties of the Fourier transform are used to show the converse. So in some sense a form of hand waving is used by blindly asking the student to accept the reversibility of the Fourier transform. In fact it took mathematicians quite a while after Fourier's initial work to make the proofs for his theory rigorous.

But this paper is going to show how these two particular functions form a Fourier pair the old fashioned way by direct integration.

So first we will lay some groundwork with some definitions:

$$Sync(x) = \frac{\sin(x)}{x}$$

and

$$\operatorname{Re}ct(x) = \begin{matrix} 1 & when & |x| < 1 \\ 0.5 & when & |x| = 1 \\ 0 & when & |x| > 1 \end{matrix}$$

Likewise, we will define the Fourier transform (direct and inverse respectively) as

Analysis: $F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

Synthesis: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$

[1] Depending on the definition of the Fourier transform, there are same scaling constants.

Now we will 1st tackle the trivial case – namely find the Fourier transform of the Rect() function.

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\,ct(x)e^{-i\omega x}dx = 2\int_{0}^{\infty} \mathrm{Re}\,ct(x)\cos(\omega x)dx = 2\int_{0}^{1}\cos(\omega x)dx = \frac{2\sin(\omega)}{\omega} = 2Sync(\omega)$$

Now to go the other way, we will find two integrals very handy:

$$\int_{0}^{\infty}\frac{a}{a^2+s^2}ds = \frac{1}{a}\int_{0}^{\infty}\frac{1}{1+y^2}\,|\,a\,|\,dy = \mathrm{sgn}(a)\left[\tan^{-1}y\right]_0^{\infty} = \mathrm{sgn}(a)\frac{\pi}{2}$$

It is easy to forget that the Jacobian requires an absolute value and then one erroneously arrives at an answer without the sgn() function.

$$\int_{0}^{\infty}\sin(ax)e^{-sx}dx = \frac{a}{a^2+s^2} \quad \text{where s is positive}$$

This latter integral is found by applying integration by parts twice.

Now for the inverse Fourier transform of the sinc() function we start with definition.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}2Sync(\omega)e^{i\omega x}d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}\left(\cos(\omega x)+i\sin(\omega x)\right)d\omega$$

Next exploit the fact that Sync() is even, so

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(\omega)\cos(\omega x)}{\omega}d\omega$$

Now use a trig identity to expand the product into a sum

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\left(\omega(1-x)\right)+\sin\left(\omega(1+x)\right)}{\omega}d\omega$$

Next we use a trick to replace 1/omega with an integral. Specifically:

$$\frac{1}{\omega} = \int_{0}^{\infty}e^{-s\omega}ds$$

So plugging this substitution in the above, we find:

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \sin\left(\omega(1-x)\right) + \sin\left(\omega(1+x)\right) d\omega \times \int_0^{\infty} e^{-s\omega} ds$$

And now by Fubini's theorem, we can write this as a double integral

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} \left[\sin\left(\omega(1-x)\right) + \sin\left(\omega(1+x)\right)\right] e^{-s\omega} d\omega ds$$

Now let's split this into the sum of two integrals:

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} \sin\left(\omega(1-x)\right) e^{-s\omega} d\omega ds + \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} \sin\left(\omega(1+x)\right) e^{-s\omega} d\omega ds$$

Now notice that these two with respect to omega) are in the form of the second example integral in this paper, so using that result, we find after integrating out omega:

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1-x}{(1-x)^2 + s^2} ds + \frac{1}{\pi}\int_0^{\infty} \frac{1+x}{(1+x)^2 + s^2} ds$$

And finally we integrate s, to find (using the first sample integral)

$$f(x) = \frac{1}{2}\left(\operatorname{sgn}(1-x) + \operatorname{sgn}(1+x)\right)$$

which using the defintion for Rect() is seen to be:

$$f(x) = \mathrm{Re}\, ct(x)$$

Thus we have shown how Sync() and Rect() form a Fourier pair using direct integration.