A “DSP” derivation of Binet’s Formula for the Fibonacci Series

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The Fibonacci series is a series where the each term in the series is the sum of the two prior terms and the 1st two terms are simply zero and one. Some may define the series as starting with one and one, but this only affects the starting index of the terms and we will start with term zero equal to zero. The series then goes as follows:

$F = \{0,1,1,2,3,5,8,13,21,...\}$

Instead of recursively computing each term in the series, there is a formula, due to Binet, that yields the nth term without having to find all of the prior terms. So we will set out to derive Binet’s formula using techniques from DSP.

DSP Oriented Derivation

So starting with z-transforms, we may write the Fibonacci recursion as

$$\frac{y(z^{-1})}{x(z^{-1})} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \qquad (1)$$

We arrive at this form by stating that a term minus the two prior terms is zero and since we want the 1st output to be zero, we delay the input by one sample. Hence the single z^{-1} in the numerator.

Since we wish to inverse z transform this function to find the time domain series, let’s expand it via partial fraction decomposition. This will make the inverse transformation a lot simpler.

So 1st we take the right hand side of (1) and multiply it by z^2 / z^2 to put it into a form amenable for partial fraction decomposition and find

$$\frac{z^{-1}}{1 - z^{-1} - z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z}{z^2 - z - 1} \qquad (2)$$

The right hand side denominator factors into a product of two linear equations (the roots are easily found via the general quadratic formula). Thus, generically we desire to find the coefficients, A and B so that

$$\frac{z}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b} \tag{3}$$

So multiply top and bottom by $(z-a)(z-b)$ and we get

$$z = A(z-b) + B(z-a) \tag{4}$$

Now to find A and B we just let z take on each root's value in turn

For $z = a$, we find (4) says $A = \frac{a}{a-b}$ (5)

And for $z = b$, (4) says $B = \frac{-b}{a-b}$ (6)

So plugging (5) and (6) into (3), we find

$$\frac{z}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{a}{z-a} - \frac{b}{z-b}\right) \tag{7}$$

Now for our particular function (2), the roots are $\frac{1 \pm \sqrt{5}}{2}$, so let's name them.

$$a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2} \tag{8}$$

So using (8), (7) becomes

$$\frac{1}{a-b}\left(\frac{a}{z-a} - \frac{b}{z-b}\right) = \frac{1}{\sqrt{5}}\left(\frac{a}{z-a} - \frac{b}{z-b}\right) \tag{9}$$

Now we will multiply by z^{-1}/z^{-1} and (9) becomes

$$\frac{1}{\sqrt{5}}\left(\frac{az^{-1}}{1-az^{-1}} - \frac{bz^{-1}}{1-bz^{-1}}\right) \tag{10}$$

The two fractions in (10) are recognized as resulting from geometric series (one may also use long division to do these expansions), so upon expansion we find:

$$\frac{1}{\sqrt{5}}\left(\left(az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots\right)-\left(bz^{-1}+b^2z^{-2}+b^3z^{-3}+\ldots\right)\right) \quad (11)$$

These series expansions require $\left|az^{-1}\right|<1$ and $\left|bz^{-1}\right|<1$ which is true when $\left|z\right|>a$ (12)

After collecting like powers, (12) becomes

$$\frac{1}{\sqrt{5}}\left((a-b)z^{-1}+\left(a^2-b^2\right)z^{-2}+\left(a^3-b^3\right)z^{-3}+\ldots\right) \quad (13)$$

And we may of course add in a zeroth term (which is also zero) and we find

$$\frac{y(z^{-1})}{x(z^{-1})}=\frac{1}{\sqrt{5}}\sum_{m=0}^{\infty}\left(a^m-b^m\right)z^{-m} \quad (14)$$

Now recalling the general formula for inverse z transformation:

$$x[n]=\frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz \quad (15)$$

And substituting (14) for $X(z)$, we get

$$x[n]=\frac{1}{2\pi j}\oint_c\frac{1}{\sqrt{5}}\sum_{m=0}^{\infty}\left(a^m-b^m\right)z^{-m}z^{n-1}dz=\frac{1}{2\pi j\sqrt{5}}\sum_{m=0}^{\infty}\oint_c\left(a^m-b^m\right)z^{n-m-1}dz \quad (16)$$

The Cauchy-Goursat theorem says that all of these integrals will be zero except for the case where $n=m$ (contour "c" is a circle about the origin with radius > a), so (16) quickly reduces to

$$x[n]=\frac{\left(a^n-b^n\right)}{2\pi j\sqrt{5}}\oint_c z^{-1}dz=\frac{\left(a^n-b^n\right)}{2\pi j\sqrt{5}}2\pi j=\frac{a^n-b^n}{\sqrt{5}}=\frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}} \quad (17)$$

Which is of course Binet's formula for the *n*th term of Fibonacci's series.