Hilbert Transforms, Analytic Functions, and Analytic Signals

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Introduction:

Hilbert transforms are essential in understanding many modern modulation methods. These transforms effectively phase shift a function by 90 degrees independent of frequency. Of course practical implementations have limitations. For example, the phase shifting of a low frequency implies a long delay, which in turn implies a computational process that maintains a long history of the signal. Hilbert transforms are useful in creating signals with one sided Fourier transforms. Also the concepts of analytic functions and analytic signals will be shown to be related through Hilbert transforms.

90 Degree Phase Shifters:

We will take a spectral approach and start with an ideal 90-degree phase shifter. To do this we would like a function that will transform a sinusoid to another sinusoid with the same amplitude and frequency but simply phase shifted by 90 degrees. So recalling the trigonometric identities:

$$\begin{aligned} Cos(t-90^\circ) &= Sin(t) \\ Sin(t-90^\circ) &= -Cos(t) \end{aligned} \tag{1}$$

Then we desire our transform to do the following:

$$\begin{aligned} Cos(t) &\Rightarrow Sin(t) \\ Sin(t) &\Rightarrow -Cos(t) \end{aligned} \tag{2}$$

These two conditions will also cause a sinusoid with arbitrary phase to be shifted 90 degrees. This follows since it may be decomposed into a sine and a cosine and each of these components will be shifted by 90 degrees. To show this, start with a sinusoid of arbitrary phase, θ, and decompose it into a sine and a cosine components.

$$Cos(t+\theta) = Cos(t)Cos(\theta) - Sin(t)Sin(\theta) \tag{3}$$

Next transform (phase shift) the two components and reduce.

$$\begin{aligned} Cos(t-90^\circ)Cos(\theta) - Sin(t-90^\circ)Sin(\theta) &= Sin(t)Cos(\theta) + Cos(t)Sin(\theta) \\ &= Sin(t+\theta) \\ &= Cos(t+\theta-90^\circ) \end{aligned} \tag{4}$$

So we can see the effect of the transformation is a 90-degree phase shift regardless of the original phase.

We also desire the transform to be expressible as a linear convolution. Thus, the transform will also obey superposition. The linearity property was used above in the arbitrary phase case. By modeling our transform this way, we can then use a powerful theorem from Fourier analysis that equates convolution in one domain to multiplication in the other. So let’s look at the transform in the frequency domain. Recalling the Fourier pairs for sinusoids:

$$\begin{aligned} Cos(\omega_0 t) \quad &\Leftrightarrow \quad \left(\delta(\omega-\omega_0)+\delta(\omega+\omega_{0)}\right)/2 \\ Sin(\omega_0 t) \quad &\Leftrightarrow \quad \left(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right)/2j \end{aligned} \tag{5}$$

Then our frequency domain phase shift requirements become (after canceling out the twos and moving j over):

$$\begin{aligned} \left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\times H(\omega) &= -j\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \\ \left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]\times H(\omega) &= -j\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \end{aligned} \tag{6}$$

Recalling the Dirac Delta function is a distribution whose non-zero support is over an infinitely narrow region, we then only have to compare 4 points to solve this set of equations. It is helpful to look at positive frequencies separately from the negative ones. Doing this, we only compare two points at a time.

So for $\omega_0 = \omega > 0$, we find $\delta(0)\times H(\omega) = -j\delta(0)$ or simply $H(\omega) = -j$. Likewise for $-\omega_0 = \omega < 0$, we find $H(\omega) = j$. Finally for $\omega_0 = \omega = 0$, we find $\delta(0)\times H(0) = 0$, hence $H(0) = 0$.

After putting these cases together, we find the frequency domain solution for the 90-degree phase shifter is:

$$H(\omega) = -j \times Sgn(\omega) \tag{7}$$

The Signum function is defined as:

$$Sgn(\omega) = \begin{matrix} 1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{matrix} \tag{8}$$

Now we need the time domain version of the phase shifter so we can express the phase shifter as a convolution. This is achieved just by finding the inverse Fourier transform of $H(\omega)$.

The result is:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -j \times \mathrm{sgn}(\omega) \times e^{j\omega t} d\omega = \frac{j}{2\pi}\int_{-\infty}^{0} e^{j\omega t} d\omega - \frac{j}{2\pi}\int_{0}^{\infty} e^{j\omega t} d\omega = \frac{1}{\pi \times t} \tag{9}$$

So when we wish to phase shift a function, for example $v(t)$, by 90 degrees, we just convolve it with $1/(\pi \times t)$.

Recalling the standard integral form for the convolution of two functions, $v(t) * h(t)$.

$$u(t) = \int_{-\infty}^{\infty} v(\xi) h(t-\xi) d\xi \tag{10}$$

Since convolution is commutative, we also have:

$$u(t) = \int_{-\infty}^{\infty} h(\xi) v(t-\xi) d\xi \tag{11}$$

So in terms of convolution, our phase shifter has the following integral forms

$$v(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi)}{t-\xi} d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t-\xi)}{\xi} d\xi \tag{12}$$

Now that we have a definition for the forward Hilbert transform, we find the definition for the inverse transform. The inverse Hilbert transform may be thought of as Hilbert transforming a function three times. I.e., shifting 270 degrees is the same as shifting negative 90 degrees. Two shifts result in a 180-degree phase shift, which is simple negation. So a negative 90-degree phase shift is simply the negation of a 90-degree phase shift! So for the inverse Hilbert transform, we just convolve with $-1/(\pi \times t)$.

Hilbert Transform Definitions:

Some papers start by defining a Hilbert transform is as an Integral transform. Since we started from a phase shifter point of view, and then we modeled it as a convolution, we immediately have two representations for the direct and inverse transforms. Of course an elementary change of variable converts one representation into the other.

Given a function $u(t)$, then its Hilbert transform $v(t)$ is:

$$v(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(\xi)}{t-\xi} d\xi = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(t-\xi)}{\xi} d\xi \tag{13}$$

The inverse Hilbert transform is likewise similarly defined.

$$u(t) = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{v(\xi)}{t-\xi} d\xi = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{v(t-\xi)}{\xi} d\xi \tag{14}$$

Since the basic integrals are improper these are to be evaluated as Cauchy Principal Value (CPV) Integrals. This implies a careful limiting process, taken symmetrically about the singularity, which results in exact cancellation. This basically means (for the 1st variant of the transform):

$$P \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - t} d\xi = \underset{\substack{R\to\infty \\ \varepsilon\to\xi-t}}{Lim} \int_{-R}^{-\varepsilon} \frac{u(\xi)}{\xi - t} d\xi + \int_{\varepsilon}^{R} \frac{u(\xi)}{\xi - t} d\xi \tag{15}$$

Here epsilon is an increasingly small distance from the singularity. A similar limiting process is also used with the 2nd form.

Since integration is a linear operation, we see that Hilbert transformation is also a linear operation. This means any function expanded into a sum of sinusoids, can be easily Hilbert transformed by doing the appropriate 90-degree phase shifts on each of the components.

Unlike other types of transforms, Hilbert transforms leave the function in the same domain as the original – the Hilbert transform of a temporal function is itself temporal.

An example of evaluating a CPV integral

Let's evaluate:

$$P \int_{-\infty}^{\infty} \frac{dx}{x} \tag{16}$$

So using the limit approach, we find:

$$P \int_{-\infty}^{\infty} \frac{dx}{x} = \underset{\varepsilon\to 0}{Lim} \int_{-\infty}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{\infty} \frac{dx}{x} = \underset{\varepsilon\to 0}{Lim} \int_{\infty}^{\varepsilon} \frac{dy}{y} + \int_{\varepsilon}^{\infty} \frac{dx}{x} = \underset{\varepsilon\to 0}{Lim} \int_{\varepsilon}^{\infty} \frac{dx}{x} - \frac{dy}{y} = \underset{\varepsilon\to 0}{Lim} \int_{\varepsilon}^{\infty} 0 = 0 \tag{17}$$

A change of variable was made in the middle step, $y = -x$.

A simple example of finding a Hilbert transform via convolution:

Let's find the Hilbert transform of $u(t) = Cos(t)$. So inserting $Cos(t)$ into the 2nd form of the Hilbert transform integral, we obtain:

$$v(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{Cos(t-\xi)}{\xi} d\xi \tag{18}$$

Using the trigonometric identity for the cosine of a sum of angles, we now find:

$$v(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{Cos(t-\xi)}{\xi} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Cos(\xi)}{\xi} Cos(t) + \frac{Sin(\xi)}{\xi} Sin(t) d\xi \tag{19}$$

Now using your favorite integral table or an adroit integration technique [see appendix A], obtain the following identities.

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{Cos(\xi)}{\xi} d\xi = 0 \\ &\int_{-\infty}^{\infty} \frac{Sin(\xi)}{\xi} d\xi = \pi \end{aligned} \tag{20}$$

So now the solution is easy.

$$v(t) = \frac{1}{\pi} Cos(t) \int_{-\infty}^{\infty} \frac{Cos(\xi)}{\xi} d\xi + \frac{1}{\pi} Sin(t) \int_{-\infty}^{\infty} \frac{Sin(\xi)}{\xi} d\xi = Sin(t) \tag{21}$$

Thus we find $v(t) = Sin(t)$. Hence the Hilbert transform of $Cos(t)$ is $Sin(t)$. Likewise a similar process may be used to find the Hilbert transform of $Sin(t)$ is $-Cos(t)$. So this verifies that convolution with the kernel, $1/(\pi \times t)$, does indeed perform a 90-degree phase shift operation.

Analytic Functions

The theory of Hilbert transformation is intimately connected with complex analysis, so we will now look at a requirement for isotropic differentiation of complex functions. We will use the standard paradigm of using a complex variable composed of two real components combined in the following way:

$$z = x + jy \tag{22}$$

Here x and y are real values and *j* represents $\sqrt{-1}$. Similarly we can compose a complex valued function by combining two real valued functions (also standard):

$$f(z) = u(x,y) + j \times v(x,y) \tag{23}$$

Next we will utilize the standard definition of a derivative from real analysis and extend it to the complex case. This derivative limit is:

$$f'(z) = \underset{\Delta z \to 0}{Lim} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{24}$$

A difficulty with the definition is the result may depend on the direction that taken by Δz, hence, we would like to know under what conditions the derivative is path independent. So we will take two orthogonal paths and require the resulting limits to be the same. This will yield path independence and have a derivative definition that gives consistent results. Recall $\Delta z = \Delta x + j\Delta y$.

So we will first let $\Delta y = 0$ and move along the x-axis.

$$f'(z) = \underset{\Delta x \to 0}{Lim} \left[\frac{u(x+\Delta x, y) + j \times v(x+\Delta x, y) - u(x,y) - j \times v(x,y)}{\Delta x} \right] = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \tag{25}$$

Next will move along the y-axis and let $\Delta x = 0$.

$$f'(z) = \underset{j\Delta y \to 0}{Lim} \left[\frac{u(x, y+\Delta y) + j \times v(x, y+\Delta y) - u(x,y) - j \times v(x,y)}{j\Delta y} \right] = -j \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{26}$$

The requirement that the two limits are to be the same, results in the Cauchy-Riemann relations. These are found by setting the two limits equal to each other and equating the real and imaginary parts. The C-R relations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (27)$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

If a complex function obeys the C-R relations and has differentiable components, it is said to be analytic. A function may be analytic over just part of its domain. However a function can't be analytic at just a single point – if it is analytic at a point, then it also must be analytic in an open neighborhood around that point.

Since we composed our complex function by summing together two functions, one purely real and the other purely imaginary, it would seem that our complex function has two degrees of freedom. However, the function's being analytic creates an interesting restriction. If we recall that $x = \left(z + z^*\right)/2$ and $y = \left(z - z^*\right)/2j$, and $f(x, y) = u(x, y) + j \times v(x, y)$, then let's look at the derivative with respect to z^*. So using the chain rule, we find the following expansion:

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z^*}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z^*}\right) \qquad (28)$$

Next use

$$\frac{\partial f}{\partial u} = 1 \quad \frac{\partial f}{\partial v} = j \quad \frac{\partial x}{\partial z^*} = \frac{1}{2} \quad \frac{\partial y}{\partial z^*} = \frac{j}{2} \qquad (29)$$

So our derivative now has the form:

$$\frac{\partial f}{\partial z^*} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{j}{2} \qquad (30)$$

If the function obeys the C-R relations, then the partial derivative is simply zero! So in a sense, analytic functions are independent of z^*.

Another neat quality of analytic functions is the components, $u(x, y)$ and $v(x, y)$ each solve Laplace's equation. Specifically $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. These follow directly from differentiating the C-R relations. Starting with the C-R relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{31}$$

After differentiating the first relation with respect to x and the second relation with respect to y, we find:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \text{ and } \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \tag{32}$$

Now equating the mixed partials, we find $u_{xx} = -u_{yy}$, so we see that Laplace's equation is satisfied. A similar process can be applied to make the case for $v(x, y)$. Since both components solve Laplace's equation, their sum will also, so we have the following neat theorem for analytic functions:

$$\nabla^2 f(x, y) = 0 \tag{33}$$

In our proof of this, we assumed the equivalence of mixed partials and the existence of higher order derivatives. In advanced texts on complex analysis, these will be shown to be always true for analytic functions.

An Analytic Function Example

Example: Show e^{jz} is analytic. Using the exponent addition rule and Euler's identity, we find:

$$e^{jz} = e^{jx-y} = e^{-y}e^{jx} = e^{-y}(Cos(x) + jSin(x)) = e^{-y}Cos(x) + je^{-y}Sin(x) \tag{34}$$

So we have:

$$u(x, y) = e^{-y}Cos(x) \quad v(x, y) = e^{-y}Sin(x) \tag{35}$$

Now find the four partial derivatives:

$$\frac{\partial u}{\partial x} = -e^{-y}Sin(x) \quad \frac{\partial v}{\partial y} = -e^{-y}Sin(x) \quad \frac{\partial v}{\partial x} = e^{-y}Cos(x) \quad \frac{\partial u}{\partial y} = -e^{-y}Cos(x) \tag{36}$$

Plugging these into the C-R relations, one finds they are obeyed for all x and y. Such a function is analytic everywhere and is sometimes called an entire function.

Analytic Signals

Related to the concept of analytic functions is the idea of an analytic signal. One way to make one is by evaluating an analytic function along one of the axes. For our example function, I will use the real axis. Our above analytic function has the corresponding analytic signal:

$$f(x) = u(x,0) + j \times v(x,0) = Cos(x) + jSin(x) \quad (37)$$

The choice of axis depends on the analytic function. If the wrong choice is made, the result will not be an analytic signal. How to chose which axis will be made clear shortly.

Analytic signals have several properties that prove important in signal processing. The first property is analytic signals have one-sided Fourier transforms. A second property is analytic signals obey a generalization of Euler's identity. A third property is the analytic signal's imaginary portion is the Hilbert transform of its real portion.

Assuming our third property is true, then given an analytic signal, $\psi(t) = u(t) + j \times v(t)$, its Fourier transform is $\Psi(\omega) = U(\omega) + j \times V(\omega)$.

Next use the spectral form of the Hilbert transformation to find:

$$\Psi(\omega) = U(\omega) + j \times (-j)Sgn(\omega) \times U(\omega) = (1 + Sgn(\omega)) \times U(\omega) \quad (38)$$

And we can easily see that $\Psi(\omega) = 0$ when $\omega < 0$. So this signal has no negative frequency components. Likewise starting with the conjugate, $\psi^*(t)$, we find its spectral representation to be $\Psi(\omega) = (1 - Sgn(\omega)) \times U(\omega)$, which has no positive frequency components.

The components of analytic signals also obey a generalization of Euler's identity. For example:

$$u(t) = \frac{\psi(t) + \psi^*(t)}{2} \quad v(t) = \frac{\psi(t) - \psi^*(t)}{2j} \quad (39)$$

And of course:

$$\psi(t) = u(t) + j \times v(t) \quad \psi^*(t) = u(t) - j \times v(t) \quad (40)$$

For Euler's identity, just let $\psi(t) = e^{jwt} = Cos(\omega \times t) + j \times Sin(\omega \times t)$.

To show the Hilbert transform property of analytic signals, we will have to evaluate a not so obvious integral.

We want to evaluate the following integral over the 4-piece contour shown in figure 1:

$$\oint_C \frac{\psi(z)}{z-\tau} dz \tag{41}$$

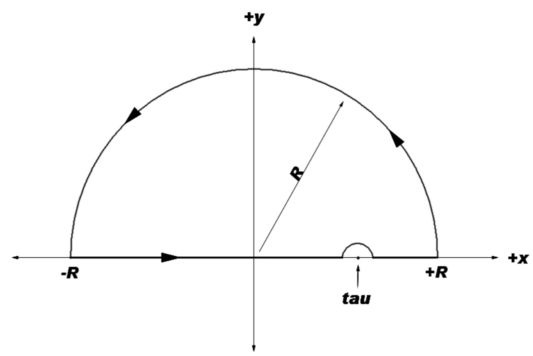

Figure 1

The four pieces are: the outer semicircle with radius R, the left horizontal line segment, the right horizontal line segment, and the inner semicircle with radius epsilon.

Since we are assuming $\psi(z)$ is analytic on and inside of c and the path takes a detour around the singularity, the Cauchy-Goursat theorem says this equals zero. So next we expand this integral into 4 path integrals whose sum is zero. Thus we have:

$$0 = \int_A \frac{\psi(z)}{z-\tau} dz + \int_{-R}^{\varepsilon} \frac{\psi(x)}{x-\tau} dx + \int_{\varepsilon}^{R} \frac{\psi(x)}{x-\tau} dx + \int_B \frac{\psi(z)}{z-\tau} dz \tag{42}$$

Since we will look at the limiting case where the limits are such that the outer semicircle becomes infinitely large and the inner semicircle likewise becomes infinitely small, we are able to combine the middle two integrals into a CPV integral. Hence,

$$0 = \int_A \frac{\psi(z)}{z-\tau} dz + P\int_{-\infty}^{\infty} \frac{\psi(x)}{x-\tau} dx + \int_B \frac{\psi(z)}{z-\tau} dz \tag{43}$$

Earlier I mentioned how the choice of axis matters. This is where we choose the axis so that Jordan's Lemma may be applied to the 1st integral to make it zero. For some analytic functions, we will use the y axis instead of the x axis. The last integral (sometimes called a detour integral since it is used to hop around the singularity evaluates to $-j\pi\psi(\tau)$. This integral is found by using a variation of Cauchy's integral formula where in the limit of the path's radius going to zero, the integral's value is $j\theta\psi(\tau)$ where θ is the angle, measured ccw, subtended by the path relative to the singularity. So plugging in these two results we find:

$$P\int_{-\infty}^{\infty}\frac{\psi(x)}{x-\tau}dx = j\pi\psi(\tau) \tag{44}$$

Now recalling $\psi(x) = u(x) + j \times v(x)$, we find:

$$P\int_{-\infty}^{\infty}\frac{u(x)}{x-\tau}dx + j\times P\int_{-\infty}^{\infty}\frac{v(x)}{x-\tau}dx = \pi(j\times u(\tau) - v(\tau)) \tag{45}$$

So by splitting this out into two real valued equations, we find:

$$\begin{aligned} v(\tau) &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{u(x)}{\tau-x}dx \\ u(\tau) &= \frac{-1}{\pi}P\int_{-\infty}^{\infty}\frac{v(x)}{\tau-x}dx \end{aligned} \tag{46}$$

These are simply the Hilbert transform relations between $u(\tau)$ and $v(\tau)$.

Appendix A: Integration tricks

Iterated Integration by Parts

Let's say we wish to integrate $\int x^2 e^{2x} dx$. Since this is an integral of a product of functions, we know that "integration by parts" would be the standard approach. However with this example, we will have to do this multiple times. There is an approach called "iterated integration by parts" that is easily applied. And the method is actually easier to remember. Basically we separate the integrand into two parts that become the headings of two columns. Each entry in the left column is formed by taking the derivative of the entry right above it. Similarly each entry in the right column is made by taking the integral of the entry right above it. Usually the columns are filled until the bottom left hand entry is zero. The integral is made up of the sum of diagonal (upper left to lower right) products with alternating sign plus the integral of the product of the entries on the bottom row taken with a continued alternating sign. If the rows are filled in until the bottom row is zero, then this last integral is not needed. An example will make this iterated process clear. For $\int x^2 e^{2x} dx$, we build the following table:

$$\begin{array}{cc} x^2 & e^{2x} \\ 2x & e^{2x}/2 \\ 2 & e^{2x}/4 \\ 0 & e^{2x}/8 \end{array} \tag{47}$$

So now combining terms with alternating sign, we find:

$$\int x^2 e^{2x} dx = +\left(x^2 \times e^{2x}/2\right) - \left(2x \times e^{2x}/4\right) + \left(2 \times e^{2x}/8\right) - \int \left(0 \times e^{2x}/8\right) dx = e^{2x}\left(x^2 - x/2 + 1/4\right) + C \tag{48}$$

So one can certainly see this method's efficacy. Since this example is one where the left hand term is a polynomial, the rows are filled in until the bottom row's product is zero. However some integrals have terms that will not go away regardless of the number of times a function is differentiated. For example let's find: $\int Sin(x) \times e^{-sx} dx$, so we build the table like this:

$$\begin{array}{cc} e^{-sx} & Sin(x) \\ -se^{-sx} & -Cos(x) \\ s^2 e^{-sx} & -Sin(x) \end{array} \tag{49}$$

So now we find:

$$\int Sin(x) e^{-sx} dx = +\left(-e^{-sx} Cos(x)\right) - \left(se^{-sx} Sin(x)\right) + \int \left(-s^2 e^{-sx} Sin(x)\right) dx + C \tag{50}$$

This is one of the cases where the integration process takes you around a loop and ends up almost where you began. Except the bottom integral works out to be an independent function times the original integral. So simplifying the algebra, we find:

$$\left(1+s^2\right)\int Sin(x)e^{-sx}dx = -e^{-sx}\left(Cos(x)+s\times Sin(x)\right)+C \tag{51}$$

Which when the integral is fully resolved yields:

$$\int Sin(x)e^{-sx}dx = \frac{-e^{-sx}\left(Cos(x)+s\times Sin(x)+C\right)}{1+s^2} \tag{52}$$

We will soon have need for the related semi-infinite definite integral, which we can easily evaluate using the above result.

$$\int_0^{\infty} Sin(x)e^{-sx}dx = \frac{1}{1+s^2} \tag{53}$$

1/x Substitution

Now let's look at a method for evaluating the integral of a sinc function. Specifically $\int_{-\infty}^{\infty}\frac{Sin(x)}{x}dx$. First realizing the integrand is even, we can reduce it to a semi-infinite interval and then perform the following tricky substitution. $\frac{1}{x}=\int_0^{\infty}e^{-st}ds$. So we can now see how to do the integral.

$$\int_{-\infty}^{\infty}\frac{Sin(x)}{x}dx = 2\int_0^{\infty}\frac{Sin(x)}{x}dx = 2\int_0^{\infty}\int_0^{\infty}e^{-sx}Sin(x)dxds = 2\int_0^{\infty}\frac{1}{1+s^2}ds = 2\times Tan^{-1}(s)\Big]_0^{\infty} = \pi \tag{54}$$