Johnson-Nyquist Noise

Clay S. Turner

Wireless Systems Engineering, Inc.
csturner@wse.biz

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In 1927, **John Bertrand Johnson** (1887-1970), conducted a series of experiments that showed electrical noise exists in all conductors and that this noise is inherent and not the result of poor component design. Johnson also provided a formula for the noise[1]. In 1928, **Harry Nyquist** (1889-1976), applied the principles of Quantum and Statisitical Mechanics and derived on a theoretical basis Johnson's formula[2]. Thermal noise as such is called *Johnson Noise* or *Johnson-Nyquist Noise*. Johnson's formula for noise power is quite simplistic.

$$P = kTb$$

Here P is the total noise power, k is Boltzmann's constant[1], T is the temperature, and b is the measurement bandwidth. It is interesting to note that the noise power in a conductor (resistor) is independent of the resistance!

Since electrical engineers like to work with power in dBm, it is useful to have Johnson's formula expressed in dBm:

$$P_{dBm} = 10\log\left(\frac{kTb}{1mW}\right) = -198.59916dBm + 10\log(Tb)$$

Which if working around room temperature $(T = 300K)$, reduces to:

$$P_{dBm} = -173.82795dBm + 10\log(b)$$

For some common resolution bandwidths available on spectrum analyzers, the room temperature Johnson formula results are tabulated below:

Temperature = 300 Kelvins	
Bandwidth (kHz)	Noise Power (dBm)
0.001	-173.82795
1	-143.82795
3	-139.05674
5	-136.83825
10	-133.82795
18	-131.27522
30	-129.05674
100	-123.82795
200	-120.81765
1000	-113.82795
1250	-112.85885

[1] The physical constants used in this paper are the 2002 CODATA values.

The cause of Johnson noise is blackbody radiation within the conductor. This explains why the noise power is independent of the resistance – it only depends on the temperature. However the resistance does affect the observed voltage. The actual charges in the conductor move so as to try to nullify the electrical fluctuations caused by the blackbody radiation. But the resistance to their movement limits the nullification, so a net random time varying potential can be observed across the conductor.

To derive Johnson's formula, we will take a modern Physics approach. Since we are talking about an ensemble of photons moving back and forth within the conductor, and photons belong to the family of bosons, the governing ensemble statistics known as the Bose-Einstein statistics tells us the expected number of particles in each state. This is given by:

$$N_n = \frac{g_n}{\exp\left[\varepsilon_n / kT\right] - 1}$$

Where the variables have the following meanings:

N_n	Number of particles in state n
g_n	Degeneracy of state n, i.e., number of ways particles have the same energy in state n. Photons have two polarizations so this equals 2.
ε_n	Energy of particle in state n. Planck's formula for photons says $\varepsilon_n = \hbar\omega_n$
k	Boltzmann's constant i.e., 1.3806505×10^{-23} J/K
T	Temperature in Kelvins
$\hbar$	Planck's constant hbar. $\hbar = \frac{h}{2\pi}$ where $h = 6.6260693\times10^{-34}$ J sec
ω	Photon's angular frequency in radians per second

Since the energy per photon is $\hbar\omega$ (Planck's Law), then multiplying this by N_n results in the total expected energy per state n. Thus after plugging in the above values for photons, the expected energy per state is:

$$\langle E_n \rangle = \frac{2\hbar\omega_n}{\exp\left[\hbar\omega_n / kT\right] - 1}$$

Now we want to find the energy density, $U(\omega)$, and this is found by $U(\omega) = E(\omega)\cdot D(\omega)\cdot d\omega$, where $D(\omega)\cdot d\omega$ is the "density of states." The density of states represents the number of allowed energy states per unit length. Now we need to find the density of states for this problem.

To do this, we will use the 1-dimensional solutions to the wave equation (The photon's wave functions are solutions to the wave equation). The solutions are of the form:

$$V_n(x,t) = V_0 \exp[i(k_n x + \omega_n t)]$$

[note: Do not confuse k_n (wave number) with k (Boltzmann's constant)]

Assuming the conductor has length L, then the boundary conditions require

$$V(0,t) = V(L,t) \rightarrow k_n L = 2n\pi \text{ where } n = 1,2,3,\ldots$$

So we find $n = \frac{L}{2\pi} k_n$ which means $\frac{dn}{dk_n} = \frac{L}{2\pi}$ (We will use this result momentarily)

The velocity of the waves (nondispersive conductor) are:

$$v = \frac{dx}{dt} = \frac{dx}{dV_n} \cdot \frac{dV_n}{dt} = \frac{\omega_n}{k_n}$$

Or we find $\frac{1}{v} = \frac{k_n}{\omega_n}$

So now we are able to find the "density of states", it is:

$$D(\omega) \cdot d\omega = \frac{1}{L} \cdot \frac{dn}{d\omega} \cdot d\omega = \frac{1}{L} \frac{dn}{dk_n} \cdot \frac{dk_n}{d\omega_n} \cdot d\omega = \frac{1}{L} \cdot \frac{L}{2\pi} \cdot \frac{1}{v} \cdot d\omega = \frac{d\omega}{2\pi v} \text{ Density of states}$$

Hence, we now find the energy density by finding $U(\omega) = E(\omega) \cdot D(\omega) \cdot d\omega$ so

$$U(\omega) = \frac{1}{2\pi v} \cdot \frac{2\hbar\omega}{\exp[\hbar\omega/kT] - 1} \cdot d\omega$$

Since we are interested in the energy flow into or out of the conductor, we just use half of the energy and multiply it by it velocity to get the power flow density.

$$P(\omega) = \frac{1}{2} v U(\omega) = \frac{1}{2\pi} \frac{\hbar\omega}{\exp[\hbar\omega/kT] - 1} d\omega$$

Next change the frequency variable from radians per second to hertz using $\omega = 2\pi f$.

$$P(f) = \frac{hf}{\exp\left[hf/kT\right] - 1} df$$

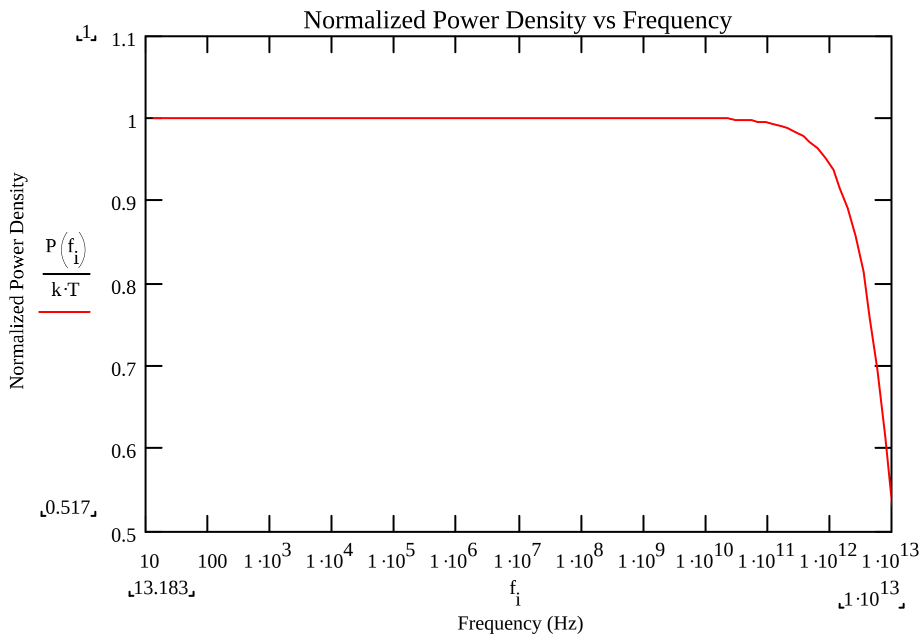
Now one may expand the exponential into a series and find:

$$P(f) = \frac{hf}{\left(\frac{hf}{kT}\right) + \left(\frac{hf}{kT}\right)^2 \left(\frac{1}{2!}\right) + \left(\frac{hf}{kT}\right)^3 \left(\frac{1}{3!}\right) + \cdots} df$$

Which after keeping terms up to 1st order yields:

$$\hat{P}(f) = kT \cdot df$$

At room temperature, this turns out to be an excellent approximation for frequencies up to tens of gigahertz. The following graph shows $P(f)/(kT)$ for T=300K. The flat part is the region where the approximation is good!



So assuming our frequencies of interest are lower than the "knee" frequency shown in the above graph, the total power (into or out of the conductor) in a bandwidth of b is:

$$\overline{P} = \int_0^b \frac{hf}{\exp\left[{hf}/{kT}\right] - 1} df \approx \int_0^b kT \cdot df = kTb \text{ Johnson's formula}$$

[1] J. B. Johnson, "Thermal Agitation of Electricity in Conductors", The American Physical Society, 1928.
[2] H. Nyquist, "Thermal Agitation of Electric Charge in Conductors", Phys. Rev. 32, 110 (1928)