Raised Cosine and Root Raised Cosine Formulae

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Introduction

The rasied cosine and root raised cosine functions (RC & RRC) are functions commonly used to avoid intersymbol interference in communications systems. They possess a couple of traits that make them useful for this purpose. Namely they are ideally bandlimited and they have periodic zeroes[1]. Plus they allow for easy selection of transition bandwidth.

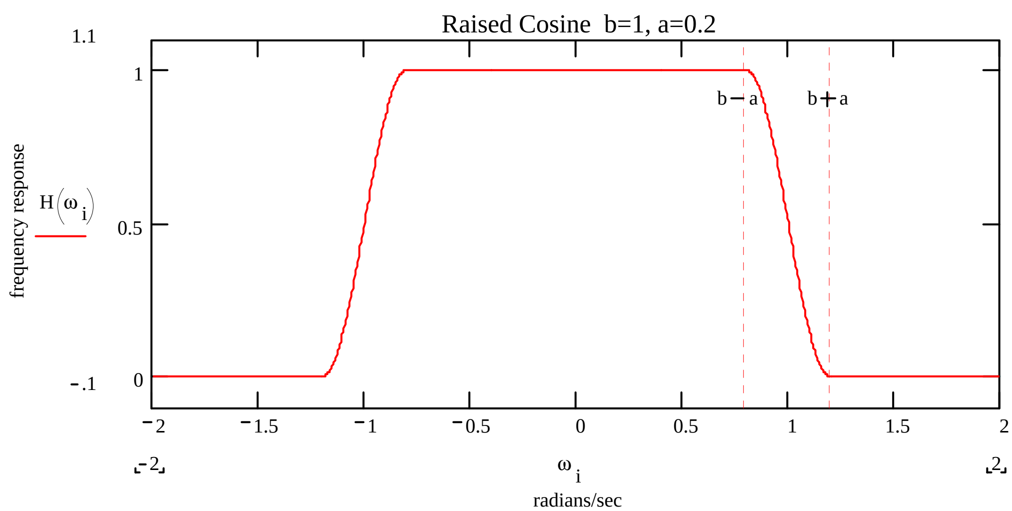
First we will start with the frequency domain representation. It is:

$$H(\omega)=\begin{cases} 1 & when \quad |\omega| \le b-a \\ \cos^p\left[\pi \dfrac{|\omega|-(b-a)}{4a}\right] & when \quad b-a \le |\omega| \le b+a \\ 0 & when \quad b+a \le |\omega| \end{cases}$$

And here the bandwidth is $2b$, the transition width is $2a$, and $p=1$ for a root raised cosine function and $p=2$ for a raised cosine function. From this definition one sees that RC and RRC are related in that $RRC(\omega)$ squared is simply $RC(\omega)$ and that they are both even functions.

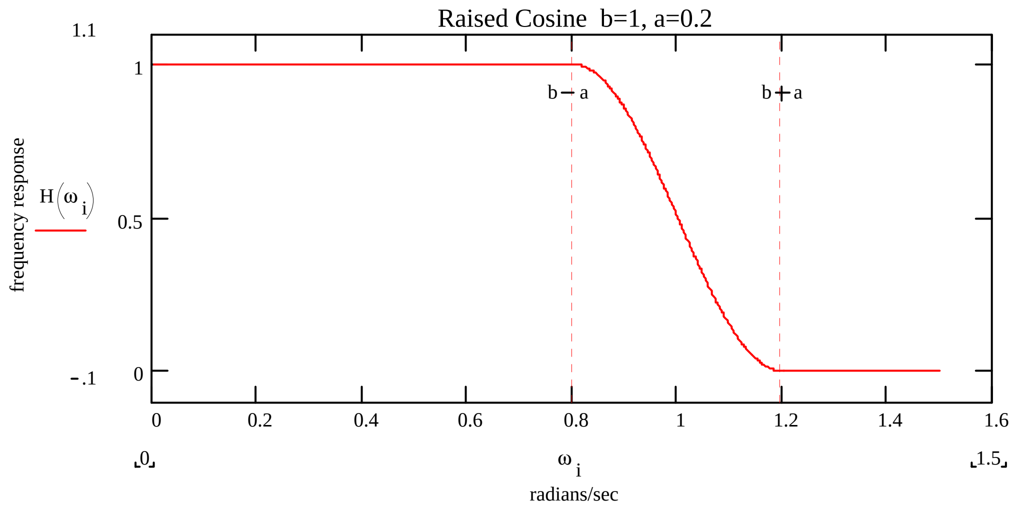
Basically the response is two constant functions, 1 and 0 joined together with a piece of a cosine (cosine squared) for the RRC and RC functions respectively. The "raised" part stems from the identity $\cos^2(x)=0.5+0.5\cos(2x)$, which says a cosine squared as being a cosine of double frequency raised up (moved vertically).

Now let's look at a graph of an RC function.

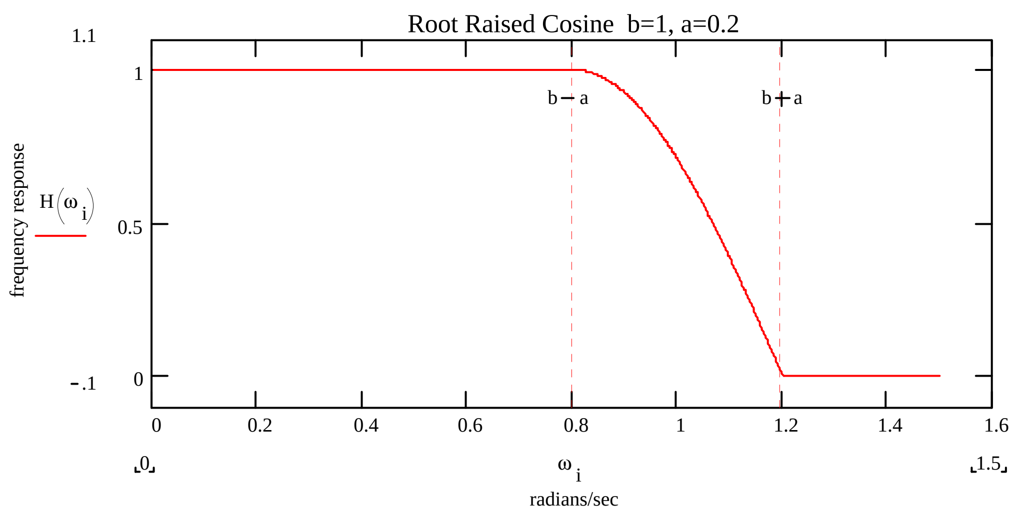


[1] While the RRC does not have strictly periodic zeroes, RRCs are to be used in pairs where the resulting RC does.

Now let’s look a little closer at the transition region for both the RC and RRC functions.



And for comparison let’s look at the same region for the RRC function.



These two graphs should make the difference obvious. The RRC function has a sharp corner at the upper edge of its transition band. And one can see how the flat parts are connected with a piece of cosine. The RRC uses 90 degrees worth and the RC uses 180 degrees worth of cosine.

Time Domain Response

Now that we’ve seen the frequency response, what most engineers need is the time domain response because that is what they will convolve their data with. This is most easily found via inverse Fourier transformation[2]. And this is where we will exploit the fact that $H(\omega)$ is even. Thus we have in general for an even function:

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[2] Often there is a scaling factor here, but since we will normalize the result, we can dispense with this.

$$h(t)=\int_{-\infty}^{\infty}H(\omega)e^{j\omega t}d\omega=2\int_{0}^{\infty}H(\omega)\cos(\omega t)d\omega$$

which after putting in the definition for RC, we have:

$$h(t)=2\int_{0}^{b-a}1\cdot\cos(\omega t)d\omega+2\int_{b-a}^{b+a}\cos^2\left[\pi\frac{\omega-(b-a)}{4a}\right]\cos(\omega t)d\omega$$

which after doing the calculus results in:

$$h(t)=2\frac{\sin(bt)}{t}\frac{\cos(at)}{1-\left(\frac{2at}{\pi}\right)^2}$$

And now we have just one more thing to do and that is to normalize the function so $h(0)=1$. For $t=0$, we find $h(0)=2b$, so we will divide by $2b$ and arrive at:

$$\hat{h}(t)=\frac{\sin(bt)}{bt}\cdot\frac{\cos(at)}{1-\left(\frac{2at}{\pi}\right)^2}\quad\text{RC impulse response}$$

Now care was taken to write the impulse response factored in such a way as to see this as a modulated sinc function. The bandwidth parameter, b, affects the rate of periodicity of zeroes and the transition width parameter, a, affects the rolloff rate. There are 3 points where simple evaluation will result in trouble. L'Hospital's rule needs to be applied to resolve these. They are at $t=0$, and at $t=\pm\pi/2a$ We already know $\hat{h}(0)=1$ due to normalization. And $\hat{h}\left(\pm\pi/2a\right)=\left(a/2b\right)\sin\left(\pi b/2a\right)$

Now to find the root raised cosine's impulse response, we do the exact same procedure. Namely we perform the following inverse Fourier transformation.

$$h(t)=2\int_{0}^{b-a}1\cdot\cos(\omega t)d\omega+2\int_{b-a}^{b+a}\cos\left[\pi\frac{\omega-(b-a)}{4a}\right]\cos(\omega t)d\omega$$

which after some calculus and normalization reslts in:

$$\hat{h}(t)=\frac{\pi^2}{\pi(a-b)-4a}\cdot\frac{4at\cos[t(a+b)]+\pi\sin[t(b-a)]}{t\left(16t^2a^2-\pi^2\right)}\quad\text{RRC impulse response}$$

And just like the RC function, there are three points where L'Hospital's rule needs to be applied. Basically at $t=0$, and $t=\pm\pi/4a$. We know $\hat{h}(0)=1$ due to normalization. And we find:

$$\hat{h}\left(\frac{\pm\pi}{4a}\right)=\frac{\pi(a+b)\sin\left(\frac{\pi(a+b)}{4a}\right)+\pi(a-b)\cos\left(\frac{\pi(b-a)}{4a}\right)-4a\cos\left(\frac{\pi(a+b)}{4a}\right)}{4a+2\pi(b-a)}$$

Discrete Time Apps – Sampled RC filters

By appropriately sampling the time domain response of the RC filter, one can easily find the coefficients to make an approximate RC filter (FIR structure). The approximation stems from the truncation of the impulse response. If we define the length of the filter to be N and the sample rate to be F_s, then we find our raised cosine filter's coefficients to be given by $RC(n) = h_1(n) \times h_2(n)$ where:

$$h_1(n) = \begin{cases} 1 & \textit{when} \quad n = \dfrac{N-1}{2} \\ \dfrac{\sin(\pi\beta(2n-N+1))}{\pi\beta(2n-N+1)} & \textit{otherwise} \end{cases}$$

and

$$h_2(n) = \begin{cases} \alpha \sin\left(\pi \dfrac{\beta}{2\alpha}\right) & \textit{when} \quad n = \dfrac{N-1}{2} \pm \dfrac{1}{4\alpha} \\ 2\beta \dfrac{\cos(\pi\alpha(2n-N+1))}{1-(2\alpha(2n-N+1))^2} & \textit{otherwise} \end{cases}$$

The frequency variables, alpha and beta are defined as follows: $\alpha = 2\pi a / f_s$ and $\beta = 2\pi b / f_s$. Of course the index n is an integer in $\{0,1,2,\cdots,N-1\}$.

By doing simple modulation[3], we can convert the RC lowpass filter into a band pass or highpass filter. For this case just simply find:

$BP(n) = h_1(n) \times h_2(n) \times \cos(\pi\gamma(2n-N+1)) \times 2$ where $\gamma = 2\pi c / f_s$ and c is the shift frequency in radians/time. The unit of time for a,b,c and f_s all need to be the same. This modulated RC filter will yield the coefficients of a frequency shifted filter. The component, $h_2(n)$, has an added factor so as to normalize the frequency response to be 1 at DC. Earlier in this paper the normalization was done to make the impulse repsonse have a peak of 1 in the time domain. But in terms of discrete filters, it is useful to have the flat top of the RC filter's response nominally have a gain of one.

[3] Multiplication by a complex exponential in the time domain is a linear shift in the frequency domain (Heaviside's theorem). In this case two shifts in opposite directions are effected by multiplying by a cosine function. Look up Euler's identity to see why this is the case.