Taylor's Series and Gain Equation Approximations

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Taylor's Series

Taylor's series result when a polynomial is made to match a function. The "matching" is the equating of the function and its derivatives to that of the polynomial at the point of expansion. When the expansion point is at the origin, then the series is a special case known as Maclaurin's series. Often the infinite series is truncated and used as an approximation to an otherwise more complicated and computationally more expensive function.

The Taylor's series is: $f(x)=\sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}\left[\left. f^{(n)}(x)\right|_{x=x_0}\right]$

From the definition one immediately "sees" that $f(x)$ and all of its derivatives must exist at point x_0. Also there are concerns with the region of convergence which standard tests are available to see if the series is viable, e.g. the limit ratio test.

Gain Functions

Sometimes DSP algorithms contain a mathematical component that serves as a gain function. And this gain function may contain an element that is computationally expensive such as finding a root or performing a division. Sometimes two or more such elements are present. Also in some applications the gain is nominally constant, so a low order Taylor's expansion can provide an efficacious approximation to the gain function.

For example, a quadrature feedback oscillator has two orthogonal outputs that when combined (as a vector) should have a constant modulus. I.e., if the outputs are $I(t)$ and $Q(t)$, then it is desired that $\sqrt{I^2(t)+Q^2(t)}=const$.

In a discrete system, we can talk about our *n*th output being simply the vector $[I_n \quad Q_n]^T$.

Also since oscillators utilize feedback, then if the modulus for the output vector is not the desired value, then the vector may be scaled so as to make it have the correct value. Thus if we define m_0 as the desired modulus and m as the measured modulus, then the gain function is simply $m_0=G\cdot m$. We will now drop the subscript n, realizing that all values used in the calculation are for the same temporal coordinate.

Now after recalling that $m = [I \quad Q][I \quad Q]^T$ and $m_0 = [\hat{I} \quad \hat{Q}][\hat{I} \quad \hat{Q}]^T$ then the gain requirement says $\begin{bmatrix} \hat{I} \\ \hat{Q} \end{bmatrix} = \sqrt{G}\begin{bmatrix} I \\ Q \end{bmatrix}$ where $G = \sqrt{\frac{m_0}{m}} = \sqrt{\frac{m_0}{I^2 + Q^2}}$.

So now let's expand $G(m)$ into a Taylor's series. This yields:

$$G(m) = 1 - \frac{(m - m_0)}{2m_0} + \frac{3}{8}\frac{(m - m_0)^2}{m_0^2} - \frac{5}{16}\frac{(m - m_0)^3}{m_0^3} + \frac{35}{128}\frac{(m - m_0)^4}{m_0^4} - \cdots$$

For two common values for the modulus, the following 1st order approximations yield:

Modulus	Gain Equation (1st order Approximation)
$1/2$	$3/2 - (I^2 + Q^2)$
1	$\frac{3 - (I^2 + Q^2)}{2}$

Another common application where a gain function shows up is in one form of an FM demodulator. If we have an analytic signal $f(t) = I(t) + jQ(t)$, then the demodulation (instantaneous frequency) is found by:

$$\omega(t) = \frac{d}{dt}\arg(f(t)) = \frac{d}{dt}\tan^{-1}\left(\frac{Q(t)}{I(t)}\right) = \frac{I(t)Q'(t) - Q(t)I'(t)}{I^2(t) + Q^2(t)}$$

So we immediately see we have a gain function of $G = 1/m$ where $m = I^2 + Q^2$. The Taylor's series is:

$$G(m) = \frac{1}{m_0} - \frac{(m - m_0)}{m_0^2} + \frac{(m - m_0)^2}{m_0^3} - \frac{(m - m_0)^3}{m_0^4} + \frac{(m - m_0)^4}{m_0^5} - \cdots$$

In a radio application where an AGC precedes the demodulation step, one expects the modulus to be somewhat constant, so depending on the effectiveness of the AGC, then only a low order approximation is needed and this may prove to be computationally economical relative to the direct division.

For example, let's say the AGC sets the nominal modulus to be 1.0. Then the 1st order approximation is:

$G(m) = 2 - (I^2 + Q^2)$ However we see that the advantage here is not as great as with the oscillator stabilization. With demodulation we are bypassing a division whereas with the oscillator we are bypassing a division and a square root!