Digital Resonators

“Discrete-Time Oscillator Theory”

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Discrete-Time Sinusoidal Resonators

- DTSRs may be represented with 2 by 2 real valued matrices.
- DTSRs are easily analyzed using matrix theory.
- DTSRs are compactly represented by matrices.
- DTSR properties are easily gleaned from the matrix representations.

Trigonometric Recursion

- Early example from Francois Vieta (1572). Vieta used this formula to recursively generate trig tables.

$$\cos(p+q) = 2\cos(p)\cdot\cos(q) - \cos(p-q)$$

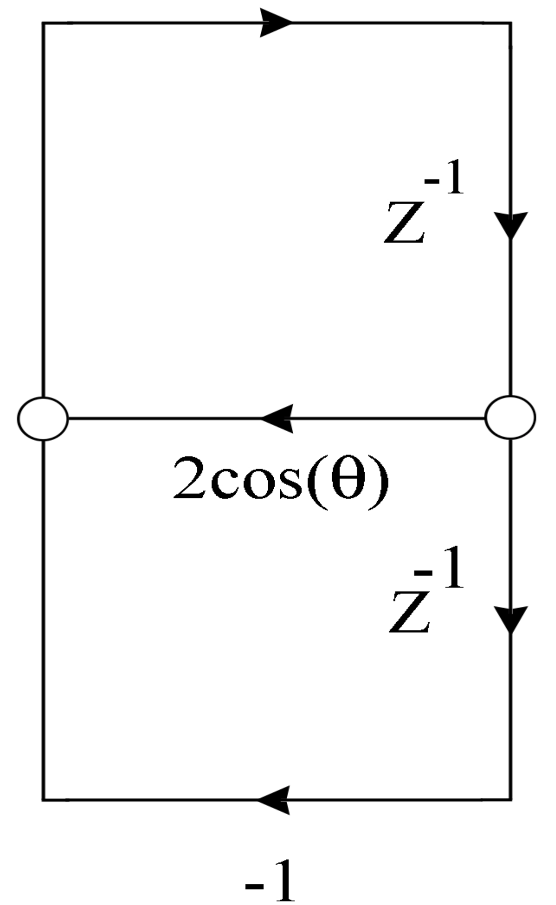
Trigonometric Recursion

- Vieta's formula allow's one to recursively find subsequent values of cos() and sin() just by using the last two values and a multiplicative factor. E.g.,

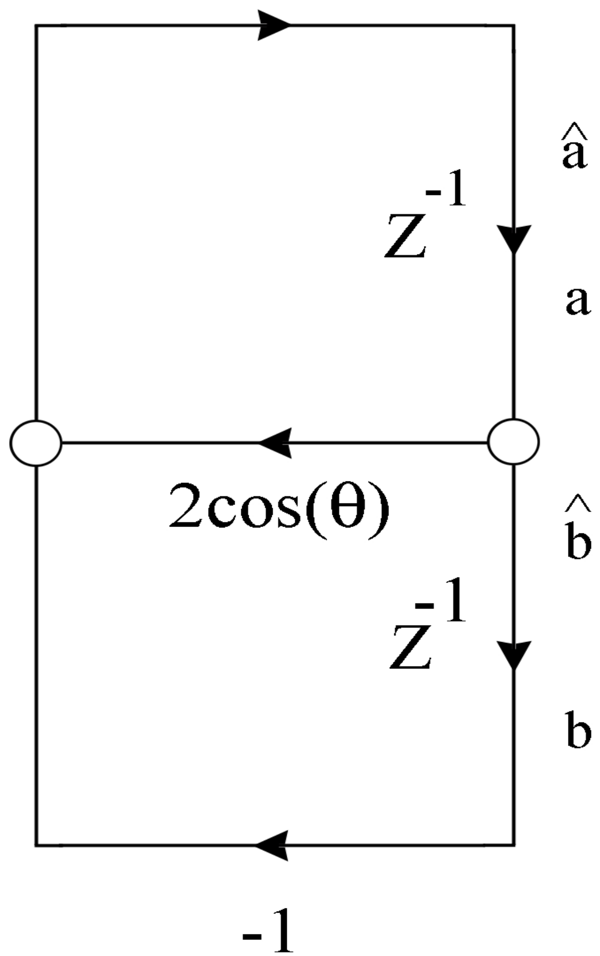
$$Next = 2\cos(p) \cdot Current - Last$$

Network to Matrix Formulation

- Network form of Vieta's recursion (Biquad).
- Biquad has two state variables (memory elements)
- Biquad uses one multiply and one subtract per iteration
- Biquad is most well known sinusoidal recursion.

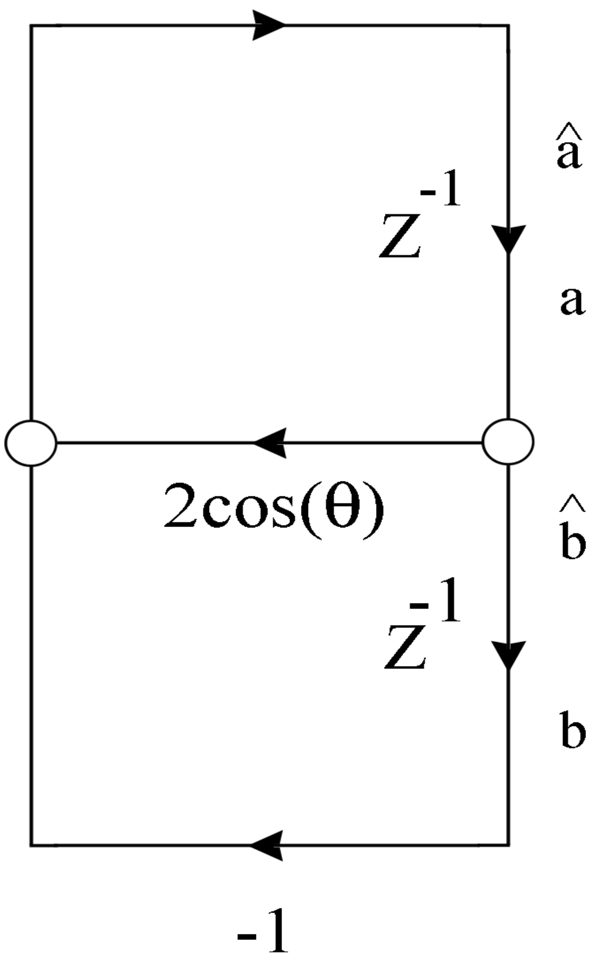


Network to Matrix Formulation



- Label inputs and the outputs of each memory element in the network.
- Then write each input's equation in terms of all outputs.

Biquad Iteration Equations



$$\hat{a} = 2\cos(\theta) - b$$

$$\hat{b} = a$$

Biquad Matrix Formulation

- Matrix Formulation of Vieta's Recursion

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 2\cos(\theta) & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

Another Trigonometric Example

- Example with two coupled equations

$$\cos(p+q)=\cos(p)\cdot\cos(q)-\sin(p)\cdot\sin(q)$$
$$\sin(p+q)=\sin(p)\cdot\cos(q)+\cos(p)\cdot\sin(q)$$

Coupled Recursion

- Here we update both sin() and cos() just using last two values and two multiplies per iteration.

$$NxtCos = CrntCos \cdot \cos(q) - CrntSin \cdot \sin(q)$$

$$NxtSin = CrntSin \cdot \cos(q) + CrntCos \cdot \sin(q)$$

Coupled Iteration Equations

- Just like with the biquad structure, we may write the iteration equations in a similar manner.
- The step angle per iteration is θ

$$\hat{a} = a \cdot \cos(\theta) - b \cdot \sin(\theta)$$

$$\hat{b} = b \cdot \cos(\theta) + a \cdot \sin(\theta)$$

Coupled Matrix Formulation

- Recognized as standard rotation matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

Discrete-Time Oscillator Structure

- The two example oscillators have the following in common:
- Each uses 2 state variables.
- The matrix in each case is 2 by 2.

Discrete-Time Linear Oscillators

- So we will look at the following types of oscillator iterations (based on a general 2 by 2 matrix) more closely:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

Discrete-Time Barkhausen Criteria

- From the point of view of repeated iteration we see the nth output of the oscillator is simply:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}_n = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^n \cdot \begin{bmatrix} a \\ b \end{bmatrix}_0$$

Resonator Theory

- Besides looking at known trigonometric relations, how do we find new oscillator structures?
- Answer: We develop our own theory with guidance from analog theory.

Barkhausen's Linear Oscillator Theory

- Heinrich Barkhausen modeled a linear oscillator as a linear amplifier with its output fed back in to its input and then stated two necessary criteria for oscillation.

Barkhausen Criteria

1. The amplifier gain times the feed back gain needs to equal unity.
2. The round trip delay needs to be a integral multiple of the oscillation period.

Discrete Time Osc. Theory

- So combining the matrix representation idea from our two examples with Barkhausen’s oscillator theory, we can come up with the discrete time counterparts to the Barkhausen Criteria.

Discrete Time Barkhausen Criteria

- The determinant of the oscillator matrix must be unity.
- The oscillator matrix when raised to some real power will be the identity matrix.

Discrete-Time Barkhausen Criteria

- Thus our two criteria in mathematical terms of matrix A are:

$$Det(A) = 1$$

$$A^{period} = I$$

Barkhausen's 1st Criterion

- If we think of the two state variables as components of a vector, then Barkhausen's 1st Criterion relates to length preservation. I.e., this says the length of the vector is unchanged by rotation.

Barkhausen’s 2nd criterion

- The periodicity constraint in combination with the unity gain means the oscillator matrix has complex valued eigenvalues.
- This in turn means the matrix’s “trace” has a magnitude of less than 2.

Discrete-Time Barkhausen Criteria

- Thus our two criteria may alternatively be stated in terms of the actual matrix elements:

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1$$

$$\left|\alpha_{11} + \alpha_{22}\right| < 2$$

Eigen-Theory

- A big advantage of using a matrix representation of the oscillator structure, is we can apply Linear Algebra with its tools to analyze oscillators.
- Eigentheory actually allows us to factor the matrix into a triple product where the step angle and the relative phases are readily apparent.

Oscillator Eigenvalues

- The oscillator matrix will have two eigenvalues that reside on the unit circle, and they are complex conjugates of each other.
- The eigenvalues are found to be:

$$\lambda = e^{\pm j\theta} \quad where \quad \theta = \cos^{-1}\left(\frac{\alpha_{11} + \alpha_{22}}{2}\right)$$

Oscillator Step Angle

- The angle theta is the step angle per iteration of the oscillator.
- Thus the oscillator frequency is determined wholly by the trace (delta) of the matrix!

$$\Delta = 2\cos(\theta)$$

Analysis of the off Diagonal Matrix Elements

- A study of the eigen values along with Barkhausen's criteria will show that the "off diagonal" matrix elements must obey the following:

$$\alpha_{12}\alpha_{21} < 0$$

Properties of the off Diagonal Matrix Elements

- Thus we see that neither off diagonal element may be zero!
- No oscillator matrix may be triangular.
- And we see that they must have opposite signs!

State Variables

- If the state variables are plotted as a function of time, they will each be a sinusoid, both with the same frequency, be out of phase and may have a relative amplitude not equal to one.

Relative Phase shift

- From a study of the eigenvectors, the relative phase shift between the state variables (b relative to a) is:

$$\phi = \arg\left(\frac{\alpha_{22} - \alpha_{11} + j\sqrt{4 - (\alpha_{11} + \alpha_{22})^2}}{2\alpha_{12}} \right)$$

Relative Phase Shift (alternative)

- An alternative formulation for relative phase shift is:

$$\phi = \cos^{-1}\left(\frac{\alpha_{22}-\alpha_{11}}{2\sqrt{-\alpha_{12}\alpha_{21}}}\right)+\frac{\pi}{2}\left(\operatorname{sgn}(\alpha_{12})-1\right)$$

Quadrature Criterion

- From the phase shift formula, we find that a relative phase of +-90 degrees requires the two elements making up the trace of the matrix to have the same value!
- Thus, from simply looking at an oscillator matrix we can ascertain if the oscillator is quadrature or not.

Relative Amplitude

- The relative amplitude (b to a) is given by:

$$\psi = \sqrt{\frac{-\alpha_{21}}{\alpha_{12}}}$$

Relative Amplitude

- From our earlier work on the off diagonal elements, we know that the relative amplitude will always be a positive value.

Equi-Amplitude Criterion

- The relative amplitude relation tells us that if we are to have both state variables have the same amplitude, then we simply require:

$$\alpha_{12} = -\alpha_{21}$$

Matrix Factoring

- The main point of the eigenanalysis is to factor the matrix. We find:

$$A = \begin{bmatrix} 1 & 1 \\ \psi e^{j\phi} & \psi e^{-j\phi} \end{bmatrix} \cdot \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ \psi e^{j\phi} & \psi e^{-j\phi} \end{bmatrix}^{-1}$$

Matrix Exponentiation

- The factoring makes raising the matrix to a power straight forward and reveals the nature of theta.

$$A^n = \begin{bmatrix} 1 & 1 \\ \psi e^{j\phi} & \psi e^{-j\phi} \end{bmatrix} \cdot \begin{bmatrix} e^{jn\theta} & 0 \\ 0 & e^{-jn\theta} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ \psi e^{j\phi} & \psi e^{-j\phi} \end{bmatrix}^{-1}$$

Initialization

- Assuming we have a chosen oscillator matrix, then from eigen-theory, we find the state variables need to be initialized to

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ \psi \cos(\phi) \end{bmatrix}$$

Nth output of oscillator

- From eigen-theory, we find the nth output (assuming the aforementioned initialization) to be:

$$\begin{bmatrix} a \\ b \end{bmatrix}_n = \begin{bmatrix} \cos(n\theta) \\ \psi\cos(n\theta+\phi) \end{bmatrix}$$

Oscillator Amplitude

- The amplitude squared (energy) of the oscillator is given by:

$$E = \frac{a^2 + \left(\frac{b}{\psi}\right)^2 - 2\frac{a \cdot b \cdot \cos(\phi)}{\psi}}{\sin^2(\phi)}$$

Amplitude Control

- To keep magnitude errors from growing, a simple “AGC” type of feedback may be employed. Thus a gain “G” is calculated every so often based on the energy, E, and is used to scale the state variables.

Useful Amplitude Control

- A 1st order Taylor's series expansion around the desired amplitude square works quite well. Two examples are:

$$G = \frac{3}{2} - E \qquad E_0 = \frac{1}{2}$$

$$G = \frac{3 - E}{2} \qquad E_0 = 1$$

Digital Waveguide Osc.

- Single Multiply, Quadrature Oscilator

$$A = \begin{bmatrix} k & k-1 \\ k+1 & k \end{bmatrix}$$

Dual Multiply - Quadrature

- This oscillator uses 2 multiplies per iteration, has quadrature outputs and uses staggered updating.

$$A = \begin{bmatrix} k & 1-k^2 \\ -1 & k \end{bmatrix}$$

Staggered Updating

- The matrix formulation's compactness is nice, but it implies a simultaneous updating of the state variables.
- Sequential updating may require temporary storage.

Staggered Updating

- Staggered updating is a method where one state variable is 1^{st} updated and then that updated value is used in the 2^{nd} update equation.

Staggered Update - Derivation

- We will start with a pair of staggered updated equations and force Barkhausen's criteria upon them.

$$\hat{b} = \alpha \cdot a + \beta \cdot b$$

$$\hat{a} = \gamma \cdot a + \delta \cdot \hat{b}$$

Staggered Update - Derivation

- So now we insert the 1st update into the 2nd equation and we get the following matrix form:

$$A = \begin{bmatrix} \gamma + \alpha\delta & \beta\delta \\ \alpha & \beta \end{bmatrix}$$

Staggered Update - Derivation

- Next we apply Barkhausen's 1st criterion and we find the following 3 parameter matrix

$$A = \begin{bmatrix} \frac{1}{\beta} + \alpha\delta & \beta\delta \\ \alpha & \beta \end{bmatrix}$$

EquiAmp-Staggered Update

- To be equi-amplitude, we just set the off-diagonal elements to be negatives of each other.

$$\alpha = -\beta\delta$$

EquiAmp-Staggered Update

- Thus our 2 parameter equi-amplitude staggered update oscillator has the following form.

$$A = \begin{bmatrix} \frac{1}{\beta} - \beta\delta^2 & \beta\delta \\ -\beta\delta & \beta \end{bmatrix}$$

EquiAmp-Staggered Update

- Now we can substitute some simple values for the parameters and get a couple of neat oscillators.
- First we will set:

$$\beta = 1$$

Magic Circle Algorithm

- The “magically” wonderful oscillator results:

$$\begin{bmatrix} 1-\delta^2 & \delta \\ -\delta & 1 \end{bmatrix}$$

Magic Circle Algorithm

- The Magic Circle Algorithm's update equations are the simple:

$$\hat{b} = b - \delta \cdot a$$

$$\hat{a} = a + \delta \cdot \hat{b}$$

Staggered Update Biquad

- Another choice for the parameters results in a Biquad oscillator with staggered updating. The tradeoff here is between an extra multiply verses a storage location. To obtain this form just let:

$$\beta\delta = 1$$

Staggered Update Biquad

- The Staggered update biquad's matrix form is:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \beta \end{bmatrix}$$

Staggered Update Biquad

- The update equations are:

$$\hat{b} = \beta \cdot b - a$$

$$\hat{a} = \frac{1}{\beta} \cdot \left(\hat{b} + a \right)$$

Reinsch (Staggered Update)

- If we relax the equiamplitude requirement and set 2 of the 3 parameters to unity, then we can obtain Reinsch’s formulation. For example:

$$\beta = 1 \quad \delta = 1$$

Reinsch (Staggered Update)

- Here we get a the following matrix and corresponding update equations:

$$A = \begin{bmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{bmatrix}$$

$$\hat{b} = \alpha \cdot a + b$$
$$\hat{\alpha} = a + \hat{b}$$

An Oscillator Application

- The venerable Goertzel Algorithm uses a Biquad oscillator for its calculation.
- However any oscillator may be used in the Goertzel Algorithm.
- Some oscillators will have better numerical properties than others. I.e., especially for low frequencies.

The Generalized Goertzel Algorithm

- Goertzel processes N values of data and computes a Fourier Coefficient (single frequency) for the data. We can describe this algorithm in four (five) main steps.

The Generalized Goertzel Algorithm (Step 1)

- Initialization – uses 1 datum.

$$\vec{y}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

The Generalized Goertzel Algorithm (Step 2)

- Recursive computation with all input data. “A” is the oscillator matrix. For

$$i \in \{1,2,3,\ldots,N-1\},\ \vec{y}_i = A \cdot \vec{y}_{i-1} + \begin{bmatrix} x_i \\ 0 \end{bmatrix}$$

The Generalized Goertzel Algorithm (Step 3)

- Phase Compensation

$$\vec{y}_N = A \cdot \vec{y}_{N-1}$$

The Generalized Goertzel Algorithm (Step 4)

- Calculation of the Fourier Coefficient

$$\begin{bmatrix} c_k \\ s_k \end{bmatrix} = \begin{bmatrix} 1 & -\dfrac{\cos(\phi)}{\psi} \\ 0 & \dfrac{\sin(\phi)}{\psi} \end{bmatrix} \cdot \left[\vec{y}_N\right]$$

The Generalized Goertzel Algorithm (Step 5a)

- Energy Calculation assuming steps 3 and 4 are performed.

$$E = c_k^2 + s_k^2$$

The Generalized Goertzel Algorithm (Step 5b)

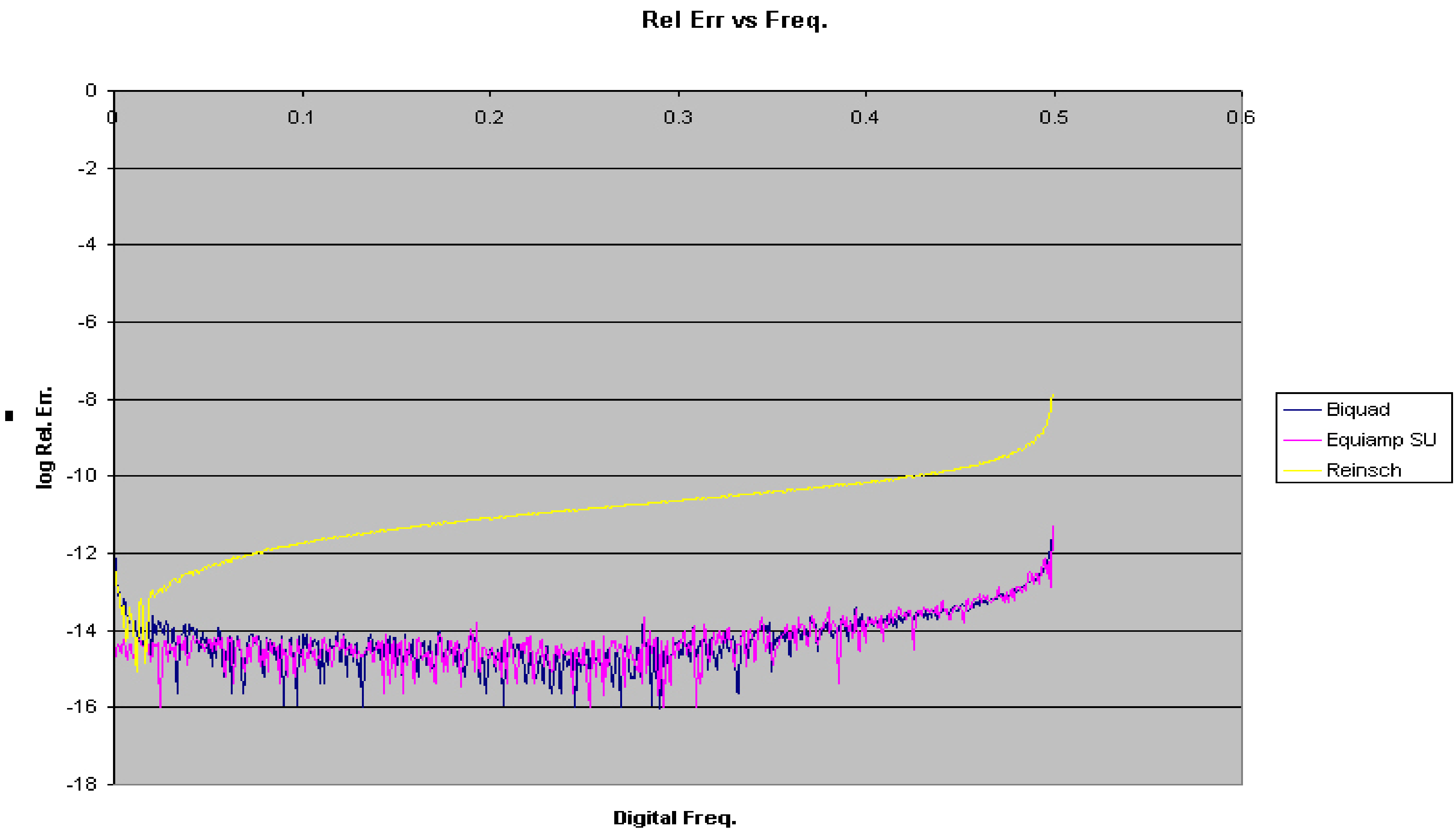
- If the energy (amplitude squared) is all that is needed, then just skip steps 3 and 4 and calculate the following: (a and b) are the 2 elements of the last result from step 2.

$$E = a^2 + \left(\frac{b}{\psi}\right)^2 - 2ab\frac{\cos(\phi)}{\psi}$$

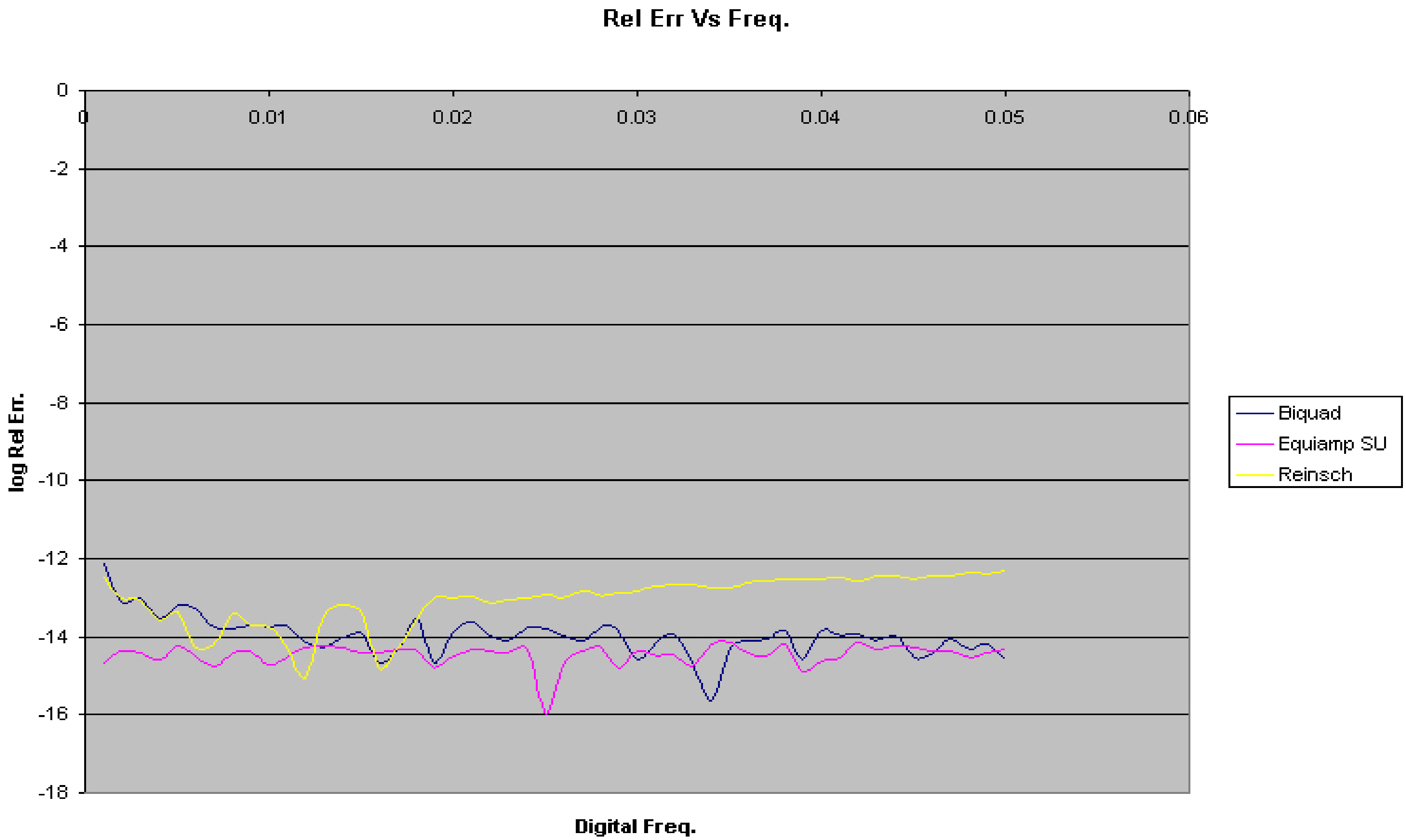
Improving Goertzel

- Some choices of oscillator design will yield better numerical accuracy than others.
- For comparison, we show the Biquad, Reinsch, and Magic Circle Oscillators and how they perform in Goertzel.
- For this test. N=1000 points

Extended Goertzel Results



Zoomed In View (Results)



Results Explaination

- In low frequency limit:
- Biquad is in phase and equiamplitude
- Reinsch is in phase and amplitudes are unmatched
- Magic Circle is quadrature and equiamplitude.

Discrete Time Oscillators

- Thanks to all who are willing to listen ;-)
- The End for Now!