Geometric Series Notes

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To simplify the notation we will work with short known length examples of geometric series and then extend to the general case. For example let's use $N = 5$ examples.

Given $S = 1 + p + p^2 + p^3 + p^4 + p^5$ then $pS = p + p^2 + p^3 + p^4 + p^5 + p^6$ and now we telescope the series by finding their difference, thus

$pS - S = p^6 - 1 \Rightarrow S(p-1) = p^6 - 1 \Rightarrow S = \dfrac{p^6 - 1}{p-1}$ Hence the general form is

$$\sum_{n=0}^{N} p^n = \frac{p^{N+1} - 1}{p - 1}$$

A related series is $S = p + 2p^2 + 3p^3 + 4p^4 + 5p^5$ which may be "summed" by differentiation. But first let's use a brute force method involving grouping of terms. Thus

$$S = (p + p^2 + p^3 + p^4 + p^5) + (p^2 + p^3 + p^4 + p^5) + (p^3 + p^4 + p^5) + (p^4 + p^5) + (p^5)$$

Now pull out the common factor from each group

$$S = p(1 + p + p^2 + p^3 + p^4) + p^2(1 + p + p^2 + p^3) + p^3(1 + p + p^2) + p^4(1 + p) + p^5(1)$$

Now using our previously found geometric series formula, we now have

$$S = p\left(\frac{p^5 - 1}{p - 1}\right) + p^2\left(\frac{p^4 - 1}{p - 1}\right) + p^3\left(\frac{p^3 - 1}{p - 1}\right) + p^4\left(\frac{p^2 - 1}{p - 1}\right) + p^5\left(\frac{p - 1}{p - 1}\right)$$

Next expand each product and sum

$$S = \frac{\left(p^6 - p\right) + \left(p^6 - p^2\right) + \left(p^6 - p^3\right) + \left(p^6 - p^4\right) + \left(p^6 - p^5\right)}{p - 1}$$

Now collect common and group together remaining terms

$$S = \frac{5p^6 - \left(p + p^2 + p^3 + p^4 + p^5\right)}{p - 1} = \frac{p}{p-1}\left(5p^5 - \left(1 + p + p^2 + p^3 + p^4\right)\right) = \frac{p}{p-1}\left(5p^5 - \left(\frac{p^5 - 1}{p - 1}\right)\right)$$

So the general form is $\displaystyle\sum_{n=0}^{N} np^n = \frac{p}{p-1}\left(Np^N - \left(\frac{p^N - 1}{p - 1}\right)\right)$

Now let's find the same result via differentiation. So starting with the identity

$$1+p+p^2+p^3+p^4+p^5=\frac{p^6-1}{p-1}$$

Differentiate both sides with respect to p. Thus

$$\frac{d}{dp}\left(1+p+p^2+p^3+p^4+p^5\right)=\frac{d}{dp}\left(\frac{p^6-1}{p-1}\right)$$

And we find

$$1+2p+3p^2+4p^3+5p^4=\frac{6p^5(p-1)-(p^6-1)}{(p-1)^2}=\frac{1}{p-1}\left(6p^5-\left(\frac{p^6-1}{p-1}\right)\right)$$

Now multiply both sides by p and we get

$$p+2p^2+3p^3+4p^4+5p^5=\frac{p}{p-1}\left(6p^5-\left(\frac{p^6-1}{p-1}\right)\right)$$

But comparing this result to the one we got earlier by grouping terms we see the results appear different, but are they really?

I.e., does $\frac{p}{p-1}\left(6p^5-\left(\frac{p^6-1}{p-1}\right)\right)=\frac{p}{p-1}\left(5p^5-\left(\frac{p^5-1}{p-1}\right)\right)$??

The issue is resolved by realizing that the inner most term on each side is a summed geometric series, so we can rewrite

$$\frac{p^6-1}{p-1}=p^5+\left(p^4+p^3+p^2+p+1\right)=p^5+\frac{p^5-1}{p-1}$$

Thus the LHS of the above becomes

$$\frac{p}{p-1}\left(6p^5-\left(\frac{p^6-1}{p-1}\right)\right)=\frac{p}{p-1}\left(6p^5-\left(p^5+\frac{p^5-1}{p-1}\right)\right)=\frac{p}{p-1}\left(5p^5-\left(\frac{p^5-1}{p-1}\right)\right)$$

So for this series we then have

$$\sum_{n=0}^{N}np^n=\frac{p}{p-1}\left((N+1)p^N-\left(\frac{p^{N+1}-1}{p-1}\right)\right)=\frac{p}{p-1}\left(Np^N-\left(\frac{p^N-1}{p-1}\right)\right)$$