Using Differential Correction to solve the GPS type Equations

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The basic problem is given 3 or more postions of satellites and their distances away, find the observation that best agrees (according to distance) with the satellite positions.

This solution uses an iterative approach based on differential correction that hails back to the days of Gauss and Legendre when finding orbital parameters when given observations.

So if a satellite's position is denoted by (X,Y,Z) and the observation position is (a,b,c) with distance D, then our required relation for each satellite (subscripts omitted for clarity) is:

$$(X-a)^2+(Y-b)^2+(Z-c)^2=D^2$$

Since we will iteratively solve for the final (a,b,c), let's do implicit differentiation with respect to D

$$-2\,(X-a)\,\frac{da}{dD}-2\,(Y-b)\,\frac{db}{dD}-2\,(Z-c)\,\frac{dc}{dD}=2\,D$$

After some algebraic simplification, we find

$$(X-a)\,da+(Y-b)\,db+(Z-c)\,dc=-DdD$$

Since we will need multiple observations, we form a system of equations

$$\begin{bmatrix} X_1-a & Y_1-b & Z_1-c \\ X_2-a & Y_2-b & Z_2-c \\ X_3-a & Y_3-b & Z_3-c \end{bmatrix}\cdot\begin{bmatrix} da \\ db \\ dc \end{bmatrix}=\begin{bmatrix} -D_1\,dD_1 \\ -D_2\,dD_2 \\ -D_3\,dD_3 \end{bmatrix}$$

We need at a minimum of 3 observations, but hopefully one has more than that. Just make the arrays as deep as needed to hold all of the observations. Also we will interpret the infinitesimal differentials as finite deltas.

To forge ahead, we let dD(s) be the difference between the measured and predicted distances for each satellite. Then we just do a least squares solution to find the da, db, and dc. Once we have these, we just subtract them from the initial a,b,c to update the observation position's value. This process gets repeated until the |da,db,dc| (differential correction) is small enough.

The following Mathcad example shows how to do this

$$S := \begin{bmatrix} 3 & .1 & 0 \\ 2 & 1.1 & 0 \\ 2 & .1 & 1 \end{bmatrix}$$

Array of satellite positions - each row is X,Y,Z

$$D := \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

Array of observed distances to satellites

$$
\begin{array}{l|l}
GPS(S,D) := & \text{for } j \in 0 .. \text{rows}(S) - 1 \\
& \quad obs\_pos_j \leftarrow 0 \\
& err \leftarrow 1 \\
& \text{while } |err| > 10^{-15} \\
& \quad \text{for } j \in 0 .. \text{rows}(S) - 1 \\
& \quad\quad \text{for } i \in 0 .. 2 \\
& \quad\quad\quad A_{j,i} \leftarrow S_{j,i} - obs\_pos_i \\
& \quad \text{for } j \in 0 .. \text{rows}(S) - 1 \\
& \quad\quad B_j \leftarrow -D_j \cdot \left[\sqrt{\displaystyle\sum_{i=0}^{2} \left(S_{j,i} - obs\_pos_i\right)^2} - D_j \right] \\
& \quad \Delta \leftarrow \left(A^T \cdot A\right)^{-1} \cdot A^T \cdot B \\
& \quad obs\_pos \leftarrow obs\_pos - \Delta \\
& \quad err \leftarrow \left(\sqrt{\Delta^T \cdot \Delta}\right)_{0,0} \\
& obs\_pos
\end{array}
$$

Init observation position to all 0, you may have a better guess.

loop until correction is very small

-D*dD

vector of da,db,dc via LSQR

update observation position

magnitude of correction

return observation position

$$GPS(S,D) = \begin{bmatrix} 2 \\ 0.1 \\ 0 \end{bmatrix}$$

Solution to example where data satellites were all on a unit sphere moved in x by 2 units and in y by 0.1 units.