Instantaneous Frequency from an Analytic Signal

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If given an analytic signal $y(t) = I(t) + jQ(t)$ then the instantaneous frequency is found by

$$f(t) = \frac{d}{dt}\arg(y(t)) = \frac{d}{dt}\tan^{-1}\left(\frac{Q(t)}{I(t)}\right) = \frac{I(t)\frac{d}{dt}Q(t) - Q(t)\frac{d}{dt}I(t)}{I^2(t) + Q^2(t)} \qquad [1]$$

For an example let $y(t) = \cos(ft) + j\sin(ft)$, then [1] yields:

$$\frac{\cos(ft) f\cos(ft) - \sin(f)(-f\sin(ft)}{\cos^2(ft) + \sin^2(ft)} = f\,\frac{\cos^2(ft) + \sin^2(ft)}{\cos^2(ft) + \sin^2(ft)} = f \qquad [2]$$

For a discrete time application one needs to find an approximation to the derivatives. One way is to replace the derivative with a difference and use a trig theorem for the difference of arctangents. Hence we find the following for the normalized frequency, f / f_s.

$$f[n] \approx \arg(y[n]) - \arg(y[n-1]) = \tan^{-1}\left(\frac{Q[n]}{I[n]}\right) - \tan^{-1}\left(\frac{Q[n-1]}{I[n-1]}\right) = \tan^{-1}\left(\frac{\frac{Q[n]}{I[n]} - \frac{Q[n-1]}{I[n-1]}}{1 + \frac{Q[n]}{I[n]}\frac{Q[n-1]}{I[n-1]}}\right)$$

$$= \tan^{-1}\left(\frac{Q[n]I[n-1] - I[n]Q[n-1]}{I[n]I[n-1] + Q[n]Q[n-1]}\right) \qquad [3]$$

For example let $y[n] = \cos(fn) + j\sin(fn)$, then [3] becomes

$$f[n] = \tan^{-1}\left(\frac{\sin(fn)\cos(fn - f) - \cos(fn)\sin(fn - f)}{\cos(fn)\cos(fn - f) + \sin(fn)\sin(fn - f)}\right)$$

$$= \tan^{-1}\left(\frac{\sin(fn)(\cos(fn)\cos(f) + \sin(fn)\sin(f)) - \cos(fn)(\sin(fn)\cos(f) - \cos(fn)\sin(f))}{\cos(fn)(\cos(fn)\cos(f) + \sin(fn)\sin(f)) + \sin(fn)(\sin(fn)\cos(f) - \cos(fn)\sin(f))}\right)$$

$$= \tan^{-1}\left(\frac{(\sin^2(fn) + \cos^2(fn))\sin(f)}{(\cos^2(fn) + \sin^2(fn))\cos(f)}\right) = \tan^{-1}\left(\frac{\sin(f)}{\cos(f)}\right) = \tan^{-1}(\tan(f)) = f$$

So it works for this signal!!