Kater's Pendulum

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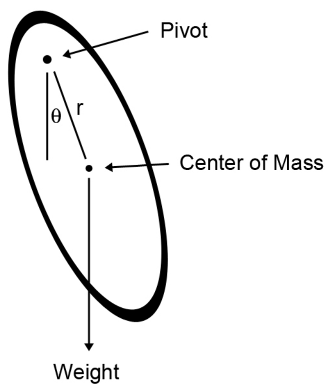
In 1817, British Physicist Henry Kater developed a reversible pendulum which allows one to precisely measure the local acceleration due to gravity. To understand what he did, we will first look at simple pendula and then we will develop Kater's result.

A pendulum is an object that is pivoted about a point (other than its center of mass) that is free to rotate about this point. Most commonly the motion is restricted to a single axis. When the pendulum's center of mass is below the pivot, the pendulum is in a equilibrium point – i.e., the pendulum's weight produces no torque and a motionless pendulum will remain motionless. If the pendulum is moved to some other position[1], the pendulum's weight produces a torque that tries to restore the pendulum to its equilibrium position. Let's refer to the following illustration.



In this diagram, the pendulum has a mass, *m*, and the center of mass is displaced from the pivot by a distance *r*. The pendulum's angular displacement is θ. To find the equation of motion, $\theta(t)$, let's 1st review torque. If a force, *f*, acts on a lever arm of length, *r*, at an angle of θ, the torque, τ, will be

$$\tau = -f \cdot r \cdot \sin(\theta)$$

The negative sign arises from the torque for a positive angular displacement (ccw) being in the negative angular direction. Also from the definition of angular acceleration, we have

$$\tau = I\alpha$$

where I is the moment of inertia about the pivot point. So now let's equate the torque expressions and fill in the missing pieces

[1] We will disregard the position where the pendulum's center of mass is directly above the pivot. While this is a stationary point, it is not stable.

$$I\alpha = -fr\sin(\theta)$$

And we know the force is simply the weight and alpha is the 2nd derivative of angular position, so we have

$$I\ddot{\theta} + mgr\sin(\theta) = 0$$

Now if we assume the angular displacement is small, then using the small angle approximation, we have

$$I\ddot{\theta} + mgr\theta = 0$$

And now slightly rearranged we have

$$\ddot{\theta} + \frac{mgr}{I}\theta = 0$$

which is the pendulum equation. A general solution is

$$\theta(t) = A\cos\left(\phi + \sqrt{\frac{mgr}{I}}t\right).$$

Thus the pendulum's motion (assuming the small angle approximation applies) is simple harmonic. The pendulum's frequency is

$\omega = \sqrt{\frac{mgr}{I}}$. And more useful is its period $T = 2\pi\sqrt{\frac{I}{mgr}}$.

Simple Pendulum

If a pendulum is modeled as a point mass, m, suspended by a massless rod of length, r, then the moment of inertia takes on a simple value. $I = mr^2$ If we plug that into the period formula, we arrive at the well known result for a simple pendulum.

$$T = 2\pi\sqrt{\frac{r}{g}}$$ Period of a simple pendulum

Kater's Approach

Kater knew that for the pendulum equation to be precise he needed to know the pendulum's moment of inertia. This amounts to knowing the mass and the radius of gyration. It is this radius that is hard to measure precisely since it depends on the distribution of the mass in the pendulum. So Kater decided to build a reversible pendulum. Essentially this is a rod with a pivot at each end and has a small movable mass located near the middle of the length of the rod. The rod is designed so that the two pivot points and the center of mass are all in a line. Let's call the two pivots "A" and "B." When the pendulum is suspended from pivot A the pendulum has one value for the period. When the pendulum is reversed and suspended from pivot B, the pendulum has another value for its period. When the movable weight is adjusted until the periods for both orientations of the pendulum are the same a special result emerges. We will now work out this special result.

First let's recall the parallel axis theorem for moments. If I have an object whose moment of inertia about its center of mass is known (let's call it I_{cm}), and it is pivoted at some displacement, *r*, from its center of mass, then the moment of inertia is $I = I_{cm} + mr^2$. So now let's equate the two periods resulting from the two orientations of the pendulum.

$$T_A = T_B$$

Let's plug in the actual period formulae

$$2\pi\sqrt{\frac{I_A}{mgr_A}} = 2\pi\sqrt{\frac{I_B}{mgr_B}}$$

And let's put in the moments of inertia

$$2\pi\sqrt{\frac{I_{cm} + mr_A^2}{mgr_A}} = 2\pi\sqrt{\frac{I_{cm} + mr_B^2}{mgr_B}}$$

And after some algebraic reduction, we arrive at the following expression for the moment of inertia about the center of mass

$$I_{cm} = mr_A r_B$$

Now let's put this into one of the period expressions (choose either one, the result will be the same!). We have

$$T_{equal} = 2\pi\sqrt{\frac{I_{cm} + mr_A^2}{mgr_A}} = 2\pi\sqrt{\frac{mr_A r_B + mr_A^2}{mgr_A}} = 2\pi\sqrt{\frac{r_B + r_A}{g}}$$

So we see Kater's remarkable result that says when a reversible pendulum has the same period in both orientations, that the pendulum behaves like a simple pendulum whose length is simply the distance between the two pivot points. Now you can see why the pivots and the center of mass need to be in a line. Otherwise $r_A + r_B$ is greater than the straight line distance between the two pivots.

Experimentally, Kater's pendulum was used to accurately measure the local acceleration of gravity. Thus one finds "g" by the following formula

$$g = (r_A + r_B)\left(\frac{2\pi}{T_{equal}}\right)^2$$

And since the individual displacements are not known – we actually know, via measurement, their sum, then our formula is actually

$$g = \ell\left(\frac{2\pi}{T_{equal}}\right)^2 \text{ where } \ell = r_A + r_B$$