Probability Computations for Some Common Games of Chance

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Apr 20, 2010 V1.1

The probabilities for many games of chance may be calculated via standard methods of counting. Thus we will first define two common formulae for this.

Combinations are enumerated via the standard formula $\binom{n}{k}$ which is shorthand notation for $\frac{n!}{(n-k)!\cdot k!}$. This represents the number of ways k items may be chosen from a set of n items without replacement. I.e., this gives us the number of ways (without regard to order) to pull k things from a bag containing n things.

The following example is for figuring the number of possible 5 card hands dealt from a 52 card deck. Thus we find $\binom{52}{5} = \frac{52!}{(52-5)!\cdot 5!} = \frac{52!}{47!\cdot 5!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2598960$.

Here I demonstrated a common computational trick of canceling out most of the terms.

The **Hypergeometric** Distribution often arises in games of chance. The distribution is denoted by $H(x,k,n,N) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$.For example, the outcome probabilities of Keno may be found by the hypergeometric distribution. Let's say the player purchases 10 numbers and wants to know the probability that 7 of them will match when the game is played. So after the player chooses his 10 numbers, then 20 numbers are drawn from a pool of 80 numbers. The probability for $x=7, k=10, N=80, n=20$ is:

$$p = \frac{\binom{10}{7}\binom{80-10}{20-7}}{\binom{80}{20}} \approx 0.001611143.$$

Video **Bingo** is a game where the player has a card, a 5 by 5 array randomly populated with 24 unique numbers from the set containing 1 to 75. The reason only 24 numbers are needed is the central square is a free spot which contains no number at all. In the video game version 50 balls are successively drawn, and if the matching spots meet certain criteria, then the player wins. One method of winning is for the "matched spots" to fill a pattern on the player's card. Commonly used patterns include: cover all 24 spots, form a picture "frame" around the edge of the card, form the letter "H" or "X", or simply to just match the 4 corners.

The find the "coverall" probability, then we simply find $H(24,24,50,75) \approx 4.71x10^{-6}$. Also for some control when designing pay tables, awards for completing the pattern in under 50 draws are permitted. So the probabilities for pattern completion when from 40 up to 50 balls are drawn have been computed for the common patterns. The table is presented below:

	Number of matched spots				
	4 Corners	X	H	Frame	Coverall
Balls Drawn	4	8	12	16	24
40	7.51903E-02	4.55838E-03	2.13860E-04	7.35108E-06	2.43814E-09
41	8.33189E-02	5.66344E-03	3.02354E-04	1.20558E-05	5.88022E-09
42	9.20893E-02	6.99602E-03	4.23295E-04	1.94747E-05	1.37205E-08
43	1.01534E-01	8.59511E-03	5.87151E-04	3.10153E-05	3.10517E-08
44	1.11688E-01	1.05051E-02	8.07333E-04	4.87383E-05	6.83137E-08
45	1.22584E-01	1.27765E-02	1.10091E-03	7.56284E-05	1.46387E-07
46	1.34259E-01	1.54663E-02	1.48946E-03	1.15964E-04	3.06081E-07
47	1.46748E-01	1.86389E-02	2.00014E-03	1.75816E-04	6.25470E-07
48	1.60089E-01	2.23667E-02	2.66685E-03	2.63724E-04	1.25094E-06
49	1.74319E-01	2.67309E-02	3.53178E-03	3.91590E-04	2.45184E-06
50	1.89477E-01	3.18225E-02	4.64707E-03	5.75867E-04	4.71508E-06

For example, the probability of getting an "H" with 46 balls are drawn is $1.489x10^{-3}$. The table is computed using the hypergeometric distribution in the following way:

$$p = \frac{\binom{x}{x}\binom{75-x}{n-x}}{\binom{75}{n}}$$ where x is the number of spots in the pattern to be matched and n is the number of balls drawn. A reduction of the hypergeometric distribution for these cases may be used since the $\binom{x}{x}$ term is simply equal to 1. Hence $p = \frac{\binom{75-x}{n-x}}{\binom{75}{n}}$

The probabilities computed here are that of getting the pattern in n balls or less and not getting the pattern when the nth ball is drawn. If you desire the probability of completing the pattern on a particular draw number or over a range of draw numbers, then we use a difference of distributions. For example what is the probability of matching 4 corners when 45 to 50 balls are drawn?

The answer is: $\frac{\binom{75-4}{50-4}}{\binom{75}{50}} - \frac{\binom{75-4}{44-4}}{\binom{75}{44}} \approx 7.78x10^{-2}$ In this case we use reduced versions of the distributions find the probability for getting 4 corners in 50 or fewer draws and subtract from that the probability of getting 4 corners in 44 or fewer draws. This gives the probability of getting 4 corners for in the range of 45 up to 50 drawn balls.

Bingo Bonus game: There are two side bets along with completing a pattern in the video Bingo. The set of 75 balls may be thought of as consisting of 5 sets of 15 balls. Thus when a player draws his 50 balls he is also seeing if he has 14 or 15 balls out of one of the 15 ball sets. The probabilities for these two cases are:

$$p_{15} = \frac{\binom{15}{15}\binom{75-15}{50-15}}{\binom{75}{50}} \approx 0.000987 \text{ and } p_{14} = \frac{\binom{15}{14}\binom{75-15}{50-14}}{\binom{75}{50}} \approx 0.01028$$

Notice how in the 1st case, we may simplify the computation and use the reduced form of the hypergeometric distribution.

Standard Bingo uses "any line" (horizontal, vertical, or diagonal) to win. Since there are many ways to place a winning line, it is simplest to let a computer program search to find all combinations. At 1st blush, this seems like one has to search 2^75 combinations. As this would be prohibitively slow, it is fortunate there is a simpler way. All we need is the probability of making any line when x numbers are on the card. Then we can use the hypergeometric distribution to split the 75 numbers into the x on the card and the 75-x that are not. Then we need to add up all probabilities for 0 up to 24 spots occupied on the card. Thus the probability, p_n, for a any complete line on a bingo card with n drawn balls is:

$$p_n = \sum_{x=0}^{24} \frac{\binom{24}{x}\binom{75-24}{n-x}}{\binom{75}{n}} \cdot p_x$$ where p_x is the probability of making any line with just x spots matched on a card. A computer program search for "any lines" results on the following data:

x	A	C	Px
0	0	1	0.000000000000000
1	0	24	0.000000000000000
2	0	276	0.000000000000000
3	0	2024	0.000000000000000
4	4	10626	0.000376435159044
5	88	42504	0.002070393374741
6	912	134596	0.006775832862789
7	5928	346104	0.017127799736495
8	27102	735471	0.036849855398785
9	92520	1307504	0.070760777787295
10	244092	1961256	0.124456980628740
11	507696	2496144	0.203392111993539
12	841100	2704156	0.311039747706863
13	1113360	2496144	0.446031959694633
14	1174620	1961256	0.598912125699042
15	981424	1307504	0.750608793548624
16	644445	735471	0.876234413049597
17	331056	346104	0.956521739130435
18	133428	134596	0.991322178965200
19	42480	42504	0.999435347261434
20	10626	10626	1.000000000000000
21	2024	2024	1.000000000000000
22	276	276	1.000000000000000
23	24	24	1.000000000000000
24	1	1	1.000000000000000

In this table, "n" is the number of spots matched on the card, "A" is the number of arrangements of "n" spots that result in any line be complete, "C" is the number of possible arrangements of "n" spots on a card. This is simply $\binom{24}{x}$ And "Px" is the probability of completing any line on a card given that "n" spots are match on the card. Our probability formula may also be written and simplified:

$$p_n = \sum_{x=0}^{24} \frac{\binom{24}{x}\binom{75-24}{n-x}}{\binom{75}{n}} \cdot \frac{A_x}{\binom{24}{x}} = \sum_{x=0}^{24} \frac{\binom{75-24}{n-x}}{\binom{75}{n}} \cdot A_x$$

And like before, this probability is for getting any line in "n" or fewer balls drawn.