Royal Flushes and Computing their Probabilities

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A standard poker deck contains 52 cards subdivided into 4 suits where each suit has 13 members. Poker hands consist of 5 cards and are named according to the arrangement of values and suits. A Royal Flush is defined as a hand of 5 cards containing 5 cards all from a single suit and spanning the values of 10 through Ace. As such a standard deck contains 4 possible Royal Flushes.

Counting

The standard approach in poker for computing the odds of various hands uses "counting" where the probability of a hand's occurrence is the number of ways the hand may be comprised divided by the number of ways 5 cards may be chosen at random from a standard deck.

Combinations

The number of ways one may pick k items from a bag containing n items without replacement is given by $\binom{n}{k}$ which is defined to be $\frac{n!}{(n-k)!k!}$. If k > n or k < 0, then the combinations evaluates to zero.

For example, the number of possible 5 card poker hands in a standard deck is $\binom{52}{5}$ which evaluates to 2598960.

Thus since a standard deck contains 4 Royal Flushes, then the probability of a Royal Flush dealt in a 5 card hand is $\frac{4}{2598960}$ which reduces to $\frac{1}{649740} \approx 0.0000015391$.

Now the situation becomes more complicated if one is allowed to discard up to all 5 of their cards and draw their replacements from the deck. This turns out to significantly increase the probability of obtaining a Royal Flush.

We are going to assume we are always trying to get a Royal Flush. The strategy used here is based upon which cards one is dealt. One examines their cards and keeps the largest number of cards (all with the same suit) which also have a value greater than or equal to 10. If you have equal numbers of cards of different suits with values greater than or equal to 10, then it doesn't matter which suit you keep. This means your initial hand will fit into one of 6 categories determined buy how many "Royal Flush" cards you should keep.

Note: This strategy is based upon the concept of always trying to get a Royal Flush. Often one does not do this, but instead determines the discards based upon the concept of maximizing the expected value of the hand and this of course depends on the pay table.

| Intermediate Royal Flush Probability Calculation | | | | | |
|--|-----------------|--------------|-------------|--------------|--|
| # of Cards Kept | # Cards to Draw | Combinations | Probability | Decimal | |
| 0 | 5 | 201376 | 1798 | 0.077483301 | |
| | | | 23205 | | |
| 1 | 4 | 1731200 | 21640 | 0.6661125989 | |
| | | | 32487 | | |
| 2 | 3 | 622200 | 305 | 0.2394034537 | |
| | | | 1274 | | |
| 3 | 2 | 43240 | 1081 | 0.0166374242 | |
| | | | 64974 | | |
| 4 | 1 | 940 | 47 | 0.0003616831 | |
| | | | 129948 | | |
| 5 | 0 | 4 | 1 | 0.0000015391 | |
| | | | 649740 | | |

Now that we have one of the above 6 hand types, then we just draw the recommended number of cards. The probability of getting the correct cards drawn from the remaining deck is $\binom{x}{x} / \binom{47}{x}$, where x is the needed number of cards to complete the Royal Flush. We just multiply this by the probability of getting to the original hand to find this pathway's probability of getting a Royal Flush. These products are enumerated in the following table.

| Final Royal Flush Probability Calculation | | | | | |
|---|---------------|-------------|--------------|--|--|
| # of Cards Kept | # Cards Drawn | Probability | Decimal | | |
| 0 | 5 | 1798 | 0.000000505 | | |
| | | 35595054495 | | | |
| 1 | 4 | 4328 | 0.0000037345 | | |
| | | 1158908751 | | | |
| 2 | 3 | 61 | 0.0000147643 | | |
| | | 4131582 | | | |
| 3 | 2 | 1 | 0.0000153908 | | |
| | | 64974 | | | |
| 4 | 1 | 1 | 0.0000076954 | | |
| | | 129948 | | | |
| 5 | 0 | 1 | 0.0000015391 | | |
| | | 649740 | | | |

Thus the probability of getting a Royal Flush by any pathway is the sum of the above probabilities. This probability is p = 0.0000431746. Inverting it yields a frequency of 1 in 23161.7581812593. Allowing discards increases the probability of a Royal Flush (assuming this strategy of always going for a Royal Flush) by a factor of approximately 28 times.

The number of combinations forming the 6 hand types listed in table 1 were calculated as follows:

Draw 5

$$\binom{32}{5} = 201376$$

Draw 4

$$4\binom{32}{4}\binom{5}{1} + 6\binom{32}{3}\binom{5}{1}^{2} + 4\binom{32}{2}\binom{5}{1}^{3} + \binom{32}{1}\binom{5}{1}^{4} = 1731200$$

Draw 3

$$\binom{5}{2}\left[4\binom{32}{3}+12\binom{5}{1}\binom{32}{2}+6\binom{5}{2}\binom{32}{1}+12\binom{5}{1}^{2}\binom{32}{1}+4\binom{5}{1}^{3}+12\binom{5}{2}\binom{5}{1}\right] = 622200$$

Draw 2

$$\binom{4}{1}\binom{5}{3}\left[\binom{32}{2}+3\left[\binom{5}{2}+\binom{5}{1}\left[\binom{5}{1}+\binom{32}{1}\right]\right]=43240$$

Draw 1

$$\binom{4}{1}\binom{5}{4}\binom{47}{1} = 940$$

Draw 0

$$\binom{4}{1}\binom{5}{5} = 4$$