Here are some notes to show how to telescope a series. For example, let's say we desire to find the closed form of the following series:

$$s = 1^2 + 2^2 + 3^2 + \dots + n^2$$

So let's use two series and combine them in such a way as to cancel out all but two terms. This is like collapsing a telescope. The advantage of this method is we will only have to work with two terms (the first and last) and not worry about all of the intervening terms. So to solve the above, we will start with:

$$(2^3 + 3^3 + 4^3 + \dots + (n+1)^3) - (1^3 + 2^3 + 3^3 + \dots + n^3)$$

Yes, the trick here uses one power higher than the final series. In a more compact notation, this is

$$\sum_{k=1}^{n} (k+1)^{3} - \sum_{k=1}^{n} k^{3} = (n+1)^{3} - 1^{3} = n^{3} + 3n^{2} + 3n^{3}$$

Starting from the same place, we also may expand the terms being summed and then reduce, thus

$$\sum_{k=1}^{n} (k+1)^{3} - \sum_{k=1}^{n} k^{3} = \sum_{k=1}^{n} (k^{3} + 3k^{2} + 3k + 1) - \sum_{k=1}^{n} k^{3}$$

We may combine this into a single summation

$$=\sum_{k=1}^{n} \left(3k^{2} + 3k + 1\right)$$

Now equate this with the earlier result for the telescoped series

$$n^{3} + 3n^{2} + 3n = \sum_{k=1}^{n} (3k^{2} + 3k + 1)$$

Rearrange to put what we desire on the left hand side

$$\sum_{k=1}^{n} 3k^{2} = n^{3} + 3n^{2} + 3n - \sum_{k=1}^{n} 3k - \sum_{k=1}^{n} 1$$

Which reduces to

$$3\sum_{k=1}^{n} k^{2} = n^{3} + 3n^{2} + 3n - 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

Now plug in the results for series of lower powers

$$3\sum_{k=1}^{n} k^{2} = n^{3} + 3n^{2} + 3n - 3\frac{n}{2}(n+1) - n$$

Divide by 3

$$\sum_{k=1}^{n} k^{2} = \frac{n^{3}}{3} + n^{2} + n - \frac{n^{2}}{2} - \frac{n}{2} - \frac{n}{3}$$

Get a common denominator of 6, combine and then factor

$$\sum_{k=1}^{n} k^2 = \frac{2n^3}{6} + \frac{6n^2}{6} + \frac{6n}{6} - \frac{3n^2}{6} - \frac{3n}{6} - \frac{2n}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

The same trick may be employed to work out the sums of higher powers; However, you will need to know the sums of the lower powers. Try this approach for a sum of the cubes.

Interestingly, you can show that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

Which means the sum of the cubes is the square of the sum of the firsts!

If we know the form of the answer, then we can use induction to prove the result. For example, let's assume the above formula is correct for "n". Now let n be bigger by one, so

$$(n+1)^3 + \sum_{k=1}^n k^3 = \left((n+1) + \sum_{k=1}^n k\right)^2$$

Let's expand the right hand side

$$\left(\left(n+1\right)+\sum_{k=1}^{n}k\right)^{2} = \left(n+1\right)^{2}+2(n+1)\sum_{k=1}^{n}k+\left(\sum_{k=1}^{n}k\right)^{2} = \left(n+1\right)^{2}+2(n+1)\frac{n}{2}(n+1)\left(\sum_{k=1}^{n}k\right)^{2}$$

Which further reduces to

$$(n+1)^3 + \left(\sum_{k=1}^n k\right)^2$$

Thus we finally have

$$(n+1)^3 + \sum_{k=1}^n k^3 = (n+1)^3 + \left(\sum_{k=1}^n k\right)^2 \Longrightarrow \sum_{k=1}^n k^3 = \left(\sum_{k=1}^n k\right)^2$$

Thus if our step for "n" is true, then our formula for "n+1" is also true regardless of "n".

Now we need to know it is at least true for one value of "n" So we choose n=1 and Immediately see that yes

$$1^3 = (1)^2$$

Hence it is proved!

C Turner 26 Jul 2012