Time Reversal and Frequency Response

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3/27/09 V2.1

Occasionally the question arises as to how a signal's frequency content is affected when the signal is time reversed. Since the frequency content of a time domain signal is given by the fourier transform of that signal, we need to look at what effects time reversal have on its fourier transform. Not too surprisingly its magnitude function is unaffected and its phase function is negated. In the discrete case this is essentially true, but an additional phase factor can show up. I will discuss this later in this paper.

Continuous Time – Real Valued Function Case

First we will look at the continuous time case, as this is pretty straight forward to work with. Instead of exploiting some special properties of the fourier transform such as its scaling and conjugation properties, we will start with just the definition of the transform. I believe this is illustrative of some common mathematical methods, and students need to be familiar with such things. Afterwards the student may employ the aforementioned theorems to shorten the derivation.

We will work with a real valued temporal function, f(t). Thus our time reversed function is f(-t). Starting with the definition of the fourier transform, we have for our original function:

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

Now considering our time reversed function, let's go find its fourier transform.

$$\int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt = ? \tag{2}$$

In order to compare (2) to (1), we will manipulate (2) so that the integral looks like the right hand side (RHS) of (1).

So let's start with the following substitution, $x = -t$. For details look up "u" substitution in a 1st year calculus book.

$$= -\int_{\infty}^{-\infty} f(x) e^{i\omega x} dx \tag{3}$$

After swapping limits – which absorbs the external negative sign

$$= \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{4}$$

Next we conjugate both inside and outside[1].

$$= \left[\int_{-\infty}^{\infty} f^*(x) e^{-i\omega x} dx \right]^* \tag{5}$$

Next apply the property that f(t) is real valued

$$= \left[\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right]^* \tag{6}$$

And referring back to (1), we finally find

$$= F^*(\omega) \tag{7}$$

Thus for our real valued function and its time reversed variant, we have the following fourier pairs:

Real Valued Signal	Fourier Transform
$f(t)$	$F(\omega)$
$f(-t)$	$F^*(\omega)$

Since the magnitude of a complex valued function, $g(t)$ is given by $\sqrt{g^*(t) \cdot g(t)}$, then it immediately follows that both $f(t)$ and $f(-t)$ have identical magnitude functions since their fourier transforms are conjugates of each other. Explicitly:

For $f(t)$, we have $|F(\omega)| = \sqrt{F^*(\omega) \cdot F(\omega)}$ (8)

For $f(-t)$, we have $|F^*(\omega)| = \sqrt{\left(F^*(\omega)\right)^* \cdot F^*(\omega)} = \sqrt{F(\omega) \cdot F^*(\omega)} = |F(\omega)|$ (9)

Thus their magnitude functions are identical.

[1] Conjugation of a product is the product of conjugates and conjugation of an exponential with a real base just conjugates the exponent itself. Thus, if the exponent is purely imaginary, then its sign simply gets swapped.

The comparison of phase functions is simply described using the argument[2] function.

For $f(t)$, $\theta(\omega) = \arg(F(\omega))$ (10)

For $f(-t)$, $\theta(\omega) = \arg(F^*(\omega)) = -\arg(F(\omega))$ (11)

So their phase functions are simply negatives of each other.

Continuous Time – Complex Valued Function Case

If instead of a real valued function, we use a complex valued one for example, $h(t)$, then we may define its reversal as $h^*(-t)$. So now we have temporal reversal and conjugation combined. This doesn't cause any issues with our real valued case as the result is consistent, but why the conjugation? Some motivation for this lies in the steps for the real valued case where we carefully disposed of, by exploiting "realness," the conjugation of a function inside of the integral so we could relate that integral back to an earlier one. But by starting with the conjugate, we no longer require our function to be real. So our result becomes more general.

If we go back to step (2) and plug in $h^*(-t)$ and follow the steps through to (7) skipping (6), we find this general fourier pair (which by the way includes our real valued case):

Complex Valued Function	Fourier Transform
$h(t)$	$H(\omega)$
$h^*(-t)$	$H^*(\omega)$

Now if we go back to (2) and again use our complex valued function but only have time reversal and no conjugation we can find another relation just with two adroit changes of variable. Specifically, $x = -t$ and $\varpi = -\omega$.

[2] The argument function, often denoted as arg() is a 4 quadrant extension to the arctangent function for complex valued arguments. In general it returns an angle in the range of 0 to 2pi. But any integral multiple of 2pi may be added to the angle. The argument function has odd parity meaning that arg(-z) = -arg(z).

$$\int_{-\infty}^{\infty} h(-t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} h(x)e^{-ix\varpi}dx = H(\varpi) = H(-\omega) \quad (12)$$

This is a simplified example (scaling = -1) of the scaling property of the fourier transform.

Now let's combine this time reversal property with the property for a time reversed conjugated function under fourier transformation and we arrive at

$$h^*(t) = h^*(-(-t)) \Leftrightarrow H^*(-\omega) \quad (13)$$

This is sometimes called the conjugation property of the fourier transform. So adding these results to our table, we find from one canonical relation four fourier pairs:

Complex Valued Function	Fourier Transform
$h(t)$	$H(\omega)$
$h^*(-t)$	$H^*(\omega)$
$h^*(t)$	$H^*(-\omega)$
$h(-t)$	$H(-\omega)$

Discrete-Time – Real Valued Function Case

As I mentioned earlier, time reversal in the discrete-time case brings with it an extra consideration. If we have a discrete time function, $f[n]$ where $n \in \{-\infty,...,-1,0,1,...,\infty\}$ and we simply replace n with –n, we will find the relations between the discrete fourier transform for both the original time domain function and its time reversed variant to have the same properties as for the continuous time case. Namely, their magnitudes will be the same and their phase functions will be negatives of each other. So what is the extra consideration I mentioned earlier?

Well let's discuss the case of an FIR filter implemented as tapped delay line. The filter's "taps" are the values of the filter's impulse response and the filter's frequency response is found via the discrete fourier transform of the impulse response. The "extra consideration" stems from the situation where one reverses the order of the taps and then wonders what is the frequency response of the new filter in terms of magnitude and phase?

In this case we are taking a finite (length=N) sequence of data indexed from 0 to N-1 and reversing it in place with the resulting sequence being indexed from 0 to N-1. This operation is more than a simple time reversal. It is a time reversal and a shift. The shift will result in a frequency dependent multiplicative phase factor in the DFT of the signal.

The justification for looking at this special case is it is common for some to analyze a finite length sequence and then if they reverse it in place and reanalyze it, to not immediately know why the phase information seems in error.

In terms of theorems one can look up the Heaviside theorem (time shifting) for fourier transforms and arrive at the proper result. Instead of that short cut, I will go back to first principles to aid students in learning basic manipulations.

So starting with a real function $f[n]$ with support over the domain $n \in \{0,1,2,...,N-1\}$, we find its discrete fourier transform (DFT) to be defined by:

$$F[k] \equiv \sum_{n=0}^{N-1} f[n] e^{-i2\pi kn/N} \tag{14}$$

Now the DFT for the time reversed and shifted signal is:

$$\sum_{n=0}^{N-1} f[N-1-n] e^{-i2\pi kn/N} = ? \tag{15}$$

Like before, we desire to manipulate (15) to be in terms of (14) so we can then compare them. So we first make the substitution $m = N-1-n$. Even though the order of summation gets reversed that doesn't matter.

$$= \sum_{m=0}^{N-1} f[m] e^{-i2\pi k(N-1-m)/N} \tag{16}$$

Next factor out the non "m" dependent part

$$= e^{-i2\pi k(N-1)/N} \sum_{m=0}^{N-1} f[m] e^{i2\pi km/N} \tag{17}$$

Now we will reduce the outside of the summation term (exploiting periodicity)

$$= e^{i2\pi k/N} \sum_{m=0}^{N-1} f[m] e^{i2\pi km/N} \tag{18}$$

Next conjugate the summation both inside and outside

$$= e^{i2\pi k/N} \left(\sum_{m=0}^{N-1} f^*[m] e^{-i2\pi km/N} \right)^* \tag{19}$$

And we know that our function is real, so

$$= e^{i2\pi k/N}\left(\sum_{m=0}^{N-1} f[m]e^{-i2\pi km/N}\right)^{*} \tag{20}$$

Which in terms of (14)

$$= e^{i2\pi k/N}F^{*}[k] \tag{21}$$

So we now have the following pairs of discrete fourier transforms.

Real Valued Signal	Discrete Fourier Transform
$f[n]$	$F[k]$
$f[N-1-n]$	$e^{i2\pi k/N}F^{*}[k]$

Starting with the magnitude function for our reversed in place signal we find it to equal the magnitude function of the original signal.

$$|\,e^{i2\pi k/N}F^{*}[k]\,| = \sqrt{\left(e^{-i2\pi k/N}F[k]\right)\left(e^{i2\pi k/N}F^{*}[k]\right)} = \sqrt{F[k]\cdot F^{*}[k]} = |\,F[k]\,| \tag{22}$$

Likewise for the phase function of the reversed in place signal, we find its relation to the phase function of the original signal.

$$\arg\left(e^{i2\pi k/N}F^{*}[k]\right) = \arg\left(e^{i2\pi k/N}\right) + \arg\left(F^{*}[k]\right) = \frac{2\pi k}{N} - \arg\left(F[k]\right) \tag{23}$$

So for the reversed in place signal we see the negation of the phase function but with the additive offset. Exploiting the periodicity of the complex exponential, we may write (23) in the following way:

$$\arg\left(e^{i2\pi k/N}e^{-i2\pi kN/N}F^{*}[k]\right) = \arg\left(e^{-i2\pi k(N-1)/N}\right) + \arg\left(F^{*}[k]\right) = -\frac{2\pi k(N-1)}{N} - \arg\left(F[k]\right) \tag{24}$$

Then we directly see that reversal in place is a time reversal (negation of the phases) and a translation towards the right by N-1 samples (the additive offset). At first blush a single left shift in (23) can’t possibly be the same as a right shift by N-1 samples or can it? To see that (23) is really the same as (24) we must account for the periodicity of the discrete fourier transform.

Shifting and Periodicity

Let's look at an example where N=5, thus our signal has five samples and its discrete fourier transform also has five values. We will denote these five values by the 1st five letters of the alphabet. Also since the DFT is periodic, we can denote the indices as 0 through 4 or we can use -4 through 0 or even -2 to 2.

In the following table we show the original data using indices of 0 through 4. With simple time reversal we show the result with indices of -4 to 0. And then we simply shift the reversed data to the right by 4 samples. This is pretty easy to follow.

But if we stay in the realm of indices from 0 through 4, the time reversal looks a little strange. It is depicted in the next to the last line below. To see this think of the indices as being numbers in the modulo class of 5. Thus -1 corresponds to 4, -2 corresponds to 3, -3 corresponds to 2 and -4 corresponds to 1. Notice how the datum at index 0 doesn't move under time reversal. Then after the time reversal with wrap around we can simply shift the data left (also with wrap around) and arrive at our reversed in place data.

Indices

-4	-3	-2	-1	0	1	2	3	4

Original Data

				a	b	c	d	e

Time Reversed (w/o wrap around)

e	d	c	b	a				

Time Reversed (w/o wrap around) and shifted right by 4 samples

e	d	c	b	a

Time Reversed (with wrap around)

a	e	d	c	b

Time Reversed and shifted left (both with wrap around)

e	d	c	b	a