A linear algebra approach to finding the phase shift and amplitude of a fourier component not exactly on a bin frequency **assuming** only one frequency is present.

by Clay S. Turner

$N := 2048$ # of points in FFT

$i := 0 .. N-1$ general index counter

$f := 82.407$

$k_1 := \frac{f}{44100} \cdot N$ ideal frequency of data $k_1 = 3.827$

$k_2 := \mathrm{floor}(0.5 + k_1)$ nearest bin frequency $k_2 = 4$

$\phi := \frac{\pi}{4}$ $\phi = 0.785$ define phase offset of real world data

$amp := 5$ define amplitude of real world data

$X_i := amp \cdot \cos\left(\frac{2 \cdot \pi \cdot i \cdot k_1}{N} + \phi\right)$ simulate real world data (single frequency and without noise)

go caculate loudest/nearest DFT bin for real world data - this comes from your FFT

Yes A and B are the compoents of your nearest loudest FFT bin to the frequency of interest.

$A := \sum_i X_i \cdot \cos\left(\frac{2 \cdot \pi \cdot k_2 \cdot i}{N}\right)$ $A = 4.625 \cdot 10^3$ real component of single bin data

$B := \sum_i X_i \cdot \sin\left(\frac{2 \cdot \pi \cdot k_2 \cdot i}{N}\right)$ $B = -1.192 \cdot 10^3$ imaginary compoent of single bin data

Matrix elements resulting from inner products between ideal and bin frequencies - note they don't depend on the data, but rather the frequencies and data length (transform size)

$$a := \sum_i \cos\left(\frac{2\cdot\pi\cdot i\cdot k_1}{N}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot i\cdot k_2}{N}\right) \qquad a = 815.712$$

$$b := \sum_i \sin\left(\frac{2\cdot\pi\cdot i\cdot k_1}{N}\right)\cdot\sin\left(\frac{2\cdot\pi\cdot i\cdot k_2}{N}\right) \qquad b = 852.311$$

$$c := \sum_i \cos\left(\frac{2\cdot\pi\cdot i\cdot k_1}{N}\right)\cdot\sin\left(\frac{2\cdot\pi\cdot i\cdot k_2}{N}\right) \qquad c = 515.055$$

$$d := \sum_i \sin\left(\frac{2\cdot\pi\cdot i\cdot k_1}{N}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot i\cdot k_2}{N}\right) \qquad d = -492.334$$

set up and solve matrix problem

$$P := \begin{bmatrix} a & -d \\ c & -b \end{bmatrix}^{-1} \cdot \begin{bmatrix} A \\ B \end{bmatrix} \qquad P = \begin{bmatrix} 3.536 \\ 3.536 \end{bmatrix}$$

The 2 componet vector 'P' is (amp*cos(phi), amp*sin(phi))

so the phase shift phi is find by using a 4 quad arctangent

$$\operatorname{atan2}\left(P_0, P_1\right) = 0.785 \qquad \phi = 0.785$$

and this finds the amplitude

$$\sqrt{\left(P_0\right)^2 + \left(P_1\right)^2} = 5$$

I hope this helps - Clay